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Dynamic meshes generation using the relaxation method with applications to fluid-structure interaction problems

Cornel Marius MUREA

University of Bucharest, Faculty of Mathematics, 14, str. Academiei, 70109 Bucharest, Romania

1 Introduction

The numerical solutions for some partial differential equations in moving boundary domains using the Finite Element Method requires to know a mesh of the domain at each time step.

We assume that the boundary of the domain is known at each instant.

We could generate the mesh of the domain at the instant $t_0 + \triangle t$ ignoring the mesh at the initial instant t_0, but this approach has the following bad points: the mesh generation takes time and we don't own a mesh generator capable to built a mesh for the 3D domains with complex geometry knowing only the "skin" of the mesh.

Consequently, we shall try to adapt the initial mesh to the current boundary.

In [1], the displacement of an interior node is computed iteratively making a mean of the displacements of the neighboring nodes.

In [3, p. 90], the displacement of an interior node is computed making a weight mean of the displacements of the boundary nodes.

In [6], the placement of the interior nodes is obtained minimizing a deformation energy of an elastic body.

Also, a method based on a minimization of a deformation energy will be employed in this paper in order to generate the dynamic meshes. The all generated meshes will have the same number of nodes as the initial mesh.

2 Presentation of the method

Let Ω^0 be a domain in $\mathbb{R}^3$ and $\mathcal{T}_h^0$ his mesh.

Let $\{A_i\}_{1\leq i\leq NBV}$ be the nodes of the mesh.

The coordinates of the nodes are $(x_i^0,\ y_i^0,\ z_i^0)_{1\leq i\leq NBV}$.

Let us consider the following sets of indices:

$$Int = \{i \in \{1, \ldots, NBV\}\,;\ A_i \text{ is an interior node of } \Omega^0\}$$
$$Fr = \{i \in \{1, \ldots, NBV\}\,;\ A_i \text{ is a boundary node of } \Omega^0\}$$

For each node A_i of the mesh, we note J_i the set of the indices of the neighboring nodes, more exactly

$$J_i = \left\{ j \in \{1, \ldots, NBV\}\,;\ [A_i A_j] \text{ is an edge of } \mathcal{T}_h^0 \right\}$$

The domain Ω^0 moves and we assume that the new coordinates $(\overline{x}_i,\ \overline{y}_i,\ \overline{z}_i)_{i \in Fr}$ of the boundary nodes are known.

The problem is to replace the interior nodes in order to obtain a reasonable mesh for the deformed domain.

Modeling each edge of the mesh by a string, the new coordinates of the interior nodes $(\overline{x}_i,\ \overline{y}_i,\ \overline{z}_i)_{i \in Int}$ will be computed minimizing the following energy:

$$J\left(x_i,\ y_i,\ z_i\ ;\ i \in Int\right) = \frac{1}{2} \sum_{\substack{1 \le i < j \le NBV \\ (i \in Int) \vee (j \in Int)}} \left[(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \right]$$

Proposition 2.1 *The optimization problem without constraints* $\inf J$ *has a unique solution characterized by:*

$$\forall i \in Int$$
$$\overline{x}_i = \frac{1}{card\,(J_i)} \sum_{j \in J_i} \overline{x}_j \quad \textit{and the similar relations for } \overline{y}_i,\ \overline{z}_i \tag{1}$$

Proof: The application J is two times continuous differentiable and her Hessian is a diagonal matrix.

Since for all i in Int we have $\partial^2 J / \partial x_i^2 = card\,(J_i)$ and the similar relations for y_i and z_i, it follows that J is elliptic (see [2], for example).

Using the standard optimization results, we obtain that the optimization problem without constraints $\inf J$ has a unique solution characterized by the relations (1). $\square$

3 Approximation using the relaxation method

In order to solve numerically the linear system (1), we could use a lot of algorithms. We have prefered a relaxation like method for her efficiency (only 3-4 iterations are sufficient for obtaining a reasonable mesh) and which can be easily implemented.

The algorithm

Step 1 We know:
$(x_i^0,\ y_i^0,\ z_i^0)_{1\le i\le NBV}$ the coordinates of the nodes of the initial mesh;
$(\overline{x}_i,\ \overline{y}_i,\ \overline{z}_i)_{i\in Fr}$ the coordinates of the boundary nodes of the moved mesh;
$\omega \in \mathbb{R}$ the relaxation parameter;
$k \leftarrow 0$ the iterations counter;

Step 2 For all $i \in Int$ we compute:

$$x_i^{k+1} = (1-\omega)\, x_i^k + \frac{\omega}{card\,(J_i)} \left(\sum_{j\in J_i \cap Int} x_j^k + \sum_{j\in J_i\cap Fr} \overline{x}_j \right)$$

We compute y_i^{k+1} and z_i^{k+1} similarly.

Step 3 We set $k \leftarrow k+1$ and go to Step 2

In the following, a convergence result will be proved.

Theorem 3.1 *For any* $(x_i^0,\ y_i^0,\ z_i^0)_{1\le i\le NBV}$ *and* $(\overline{x}_i,\ \overline{y}_i,\ \overline{z}_i)_{i\in Fr}$, *if* $\omega \in (0,1]$ *then the above algorithm is convergent to the unique solution of the optimization problem* $\inf J$.

Proof: The algorithm has the form $X^{k+1} = BX^k + c$, where

$$B = (b_{ij})_{i,j\in Int}, \quad b_{ij} = \begin{cases} 1-\omega & \text{if } i=j \\ \frac{\omega}{card(J_i)} & \text{if } j \in J_i \cap Int \\ 0 & \text{otherwise} \end{cases}$$

It is known that X^k is convergent if and only if $\rho\,(B) < 1$.
We have

$$\begin{aligned} \rho\,(B) \le \|B\|_\infty &= \max_{i\in Int} \left(\sum_{j\in J_i} |b_{ij}| \right) \\ &\le \max_{i\in Int} \left(1-\omega+\omega \frac{card\,(J_i\cap Int)}{card\,(J_i)} \right) \le 1 \end{aligned}$$

but we have to prove the strict inequality.
We proceed by contradiction: let λ be an eigenvalue of B such that $|\lambda| = 1$.
First, we prove the following inequalities:

$$\forall i \in Int, \quad |\lambda - b_{ii}| \ge \sum_{\substack{j\in Int \\ j\ne i}} |b_{ij}| \tag{2}$$

For all $i \in Int$ we have $|\lambda| - |b_{ii}| \le |\lambda - b_{ii}|$.

If there exists i in Int such that

$$|\lambda - b_{ii}| < \sum_{\substack{j \in Int \\ j \neq i}} |b_{ij}|$$

then

$$|\lambda| < \sum_{j \in Int} |b_{ij}| .$$

Since $\omega \in (0, 1]$ and using the values of b_{ij}, it follows

$$1 < 1 - \omega + \omega \frac{card\,(J_i \cap Int)}{card\,(J_i)} \leq 1$$

which is a contradiction, consequently the inequalities (2) hold.

The matrix B is evidently symmetric.

We shall prove that it is irreductible, too. It is known that a matrix is irreductible if and only if his associated graph is strongly connex (see [5, vol. 1, p. 35, Lemma 27]). The associated graph of the matrix B is in fact the mesh $\mathcal{T}_h^0$ after we have eliminated the boundary nodes and all edge that have as an end point a boundary node. Therefore B is irreductible.

Knowing that the matrix B is symmetric and irreductible, from the inequalities (2) and according to a Gerschgörin - Hadamard theorem (see [5, vol. 1, p. 57, Theorem 57]), we have

$$\forall i \in Int, \quad |\lambda - b_{ii}| = \sum_{\substack{j \in Int \\ j \neq i}} |b_{ij}|$$

It follows

$$\forall i \in Int, \quad |\lambda| - |b_{ii}| \leq \sum_{\substack{j \in Int \\ j \neq i}} |b_{ij}| \Rightarrow$$

$$\forall i \in Int, \quad 1 \leq 1 - \omega + \omega \frac{card\,(J_i \cap Int)}{card\,(J_i)}$$

But there exists $i \in Int$ such that

$$card\,(J_i \cap Int) < card\,(J_i)$$

therefore

$$\exists i \in Int, \quad 1 \leq 1 - \omega + \omega \frac{card\,(J_i \cap Int)}{card\,(J_i)} < 1$$

We have get a contradiction, consequently the theorem is proved. $\square$

4 Some numerical results

In order to made the mesh of the domain at the initial instant, we have used the 3D mesh generators MODULEF [7]. The mesh has: 74 boundary nodes, 29 interior nodes, 336 tetrahedron. All adapted meshes will have the same characteristics.

The domain moves and we can see in the Appendix A how the algorithm adapts the initial mesh to the current boundaries. Only 3-4 iterations are sufficient for obtaining the adapted meshes.

We have used NSP1B3 [4] in order to see the nodes of the mesh in a vertical section of the domain. The nodes lie at the points of the little arrows.

The initial and the current boundaries are not homothetical, therefore we can't use the homothetical transformation in order to generate the adapted meshes.

5 Applications to a fluid structure interaction problem

A three dimensional fluid structure interaction problem is studied under the following hypotheses: the fluid is incompressible and limited by an elastic structure, the all interior cavity of the structure is filled by the fluid, the structure is thick. A part of the external boundary of the structure is fixed.

We suppose that the structure is governed by the time-dependent linear elasticity equations and the fluid is governed by the time-dependent Stokes equations.

5.1 Mathematical model

Under the hypotheses above, a variational formulation is proposed in [9]. The key of this model is a Lagrange multiplier used to treat one of the problem's constraints: the equality of the fluid's and structure's velocities at the contact surface. This Lagrange multiplier has the physical signification of the density of the forces at the contact surface and it permits to decouple the problem. Knowing the density of the forces at the contact surface, we solve independently the fluid's and structure's problems and we obtain the velocity and the pressure of the fluid from the fluid's problem and the displacement and the velocity of the structure from the structure's problem.

The problem is to find the density of the forces at the contact surface such that the fluid's and structure's velocities are equal at the contact surface. The existence and the uniqueness of the solution of this three-dimensional problem are proved in [10].

5.2 Numerical aspects

In order to approximate the solution, first we had discretized in time using Finite Difference Method, after that we used Mixed-Hybrid Finite Element Method.

Discrete time problem

The time discretization corresponds to the implicit Euler method for the fluid's problem and Newmark method for the structure's problem.

In [9] it is proved that the time discrete problem is well posed. Also the stability of the semidiscrete algorithm is proved.

Finite Element approximation

In [8] it's presented the choice of the mixed-hybrid finite elements such that the fully discretized problem is well posed.

The numerical procedure proposed in the same work solves at each time step a symmetrical linear system by an iterative method. At each iteration, two decoupled problems are solved: one for the fluid and the other for the structure. The both problems have as control the density of the forces at the contact surface and the iterative method finds the "good" control such that the fluid's and structure's velocities are equal at the contact surface. The iterative method is convergent.

The stability in time of the algorithm is proved, too.

Implementation of the adapted mesh algorithm

Let us suppose that we have solved numerically the coupled fluid structure problem up to the nth time step.

Interpoling the structure velocity computed at the first n time steps, we can obtain the structure displacement at the $(n+1)^{th}$ time step.

Supposing that the all interior cavity of the structure is filled by the fluid, the new shape of the fluid domain is well established by the structure displacement. Now we adapt the initial mesh to the current boundary using the relaxation like algorithm presented in this paper and then we recompute all matrices which depend upon the new mesh of the fluid domain.

The coupled fluid structure problem at the $(n+1)^{th}$ time step was solved using the algorithm presented in [8, p. 116].

The main program calls subroutines from [4] in order to solve the fluid's problem and from [7] in order to solve the structure's problem.

In Appendix B are presented the computed velocities of the coupled fluid structure problem at two different instants. We can see that the algorithm finds the "good" density of the forces at the contact surface such that the fluid's and structure's velocities are equal at the contact surface.

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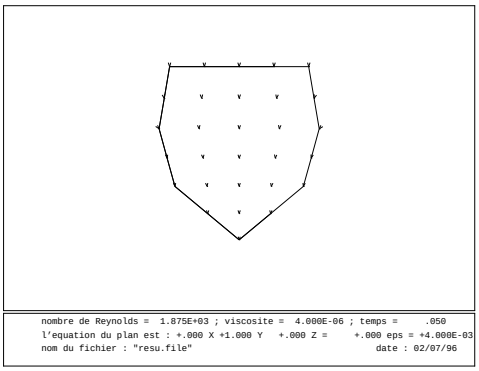
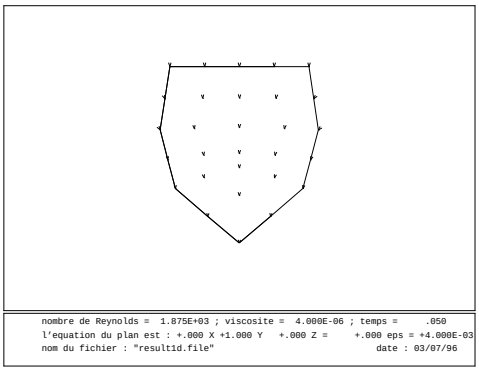
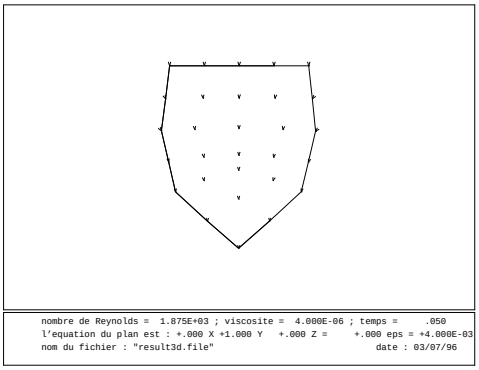
Appendix A



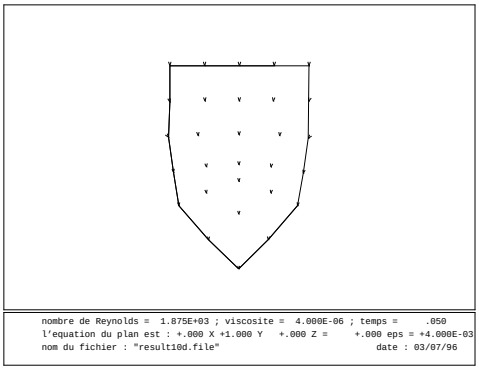
a) A vertical section of the initial mesh



b) Displacements magnitude: 2.2%



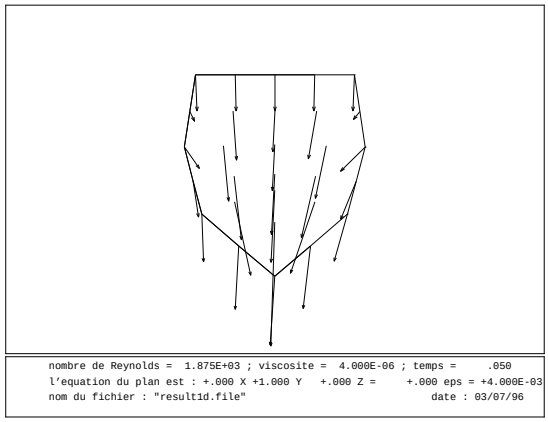
c) Displacements magnitude: 5.5%



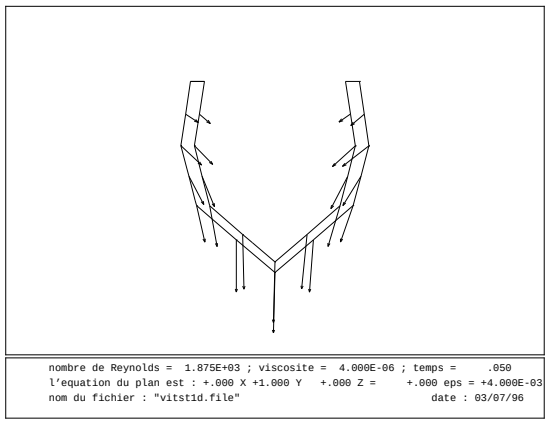
d) Displacements magnitude: 16.5%

Figure 1: The initial a) and the adapted meshes b), c), d)

Appendix B



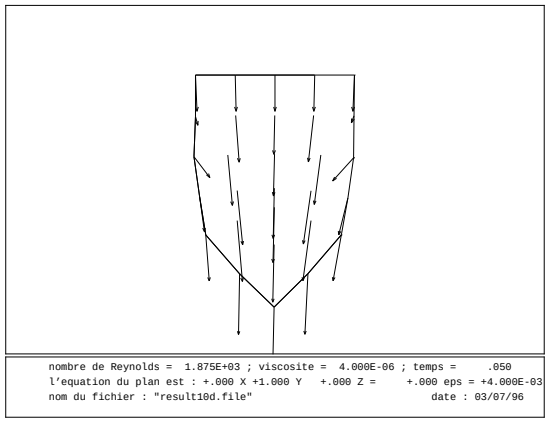
a) Computed fluid velocity



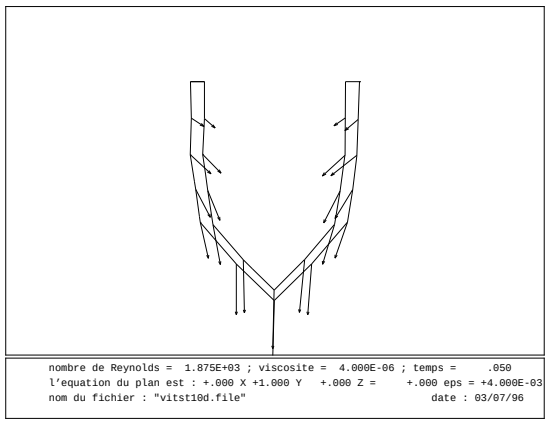
b) Computed structure velocity

Figure 2: A coupled fluid structure problem. Magnitude of the displacements: 2.2%

Appendix B (continuation)



a) Computed fluid velocity



b) Computed structure velocity

Figure 3: A coupled fluid structure problem. Magnitude of the displacements: 16.5%