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Algorithms for Optimization of the Satellite Antenna

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Abstract - The aim of this paper is to show some numerical procedures for solving a problem concerning a satellite antenna. Numerical results are presented.

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1 Introduction

We study a geostationary satellite antenna used only for emission, not for reception.

This kind of anthena is composed of many elementary sources of electro-magnetic waves. For a particular direction pointed to a station on the Earth's surface, the global electro-magnetic field must be borned by the given minimal and maximal gouges. Each elementary source is characterized by two physical parameters.

The problem is to set the both parameters of each elementary sources, such that the global field on the Earth's surface for all directions respect the gauge constraints.

Let us introduce the following notations:
ns = no. of sources, quad nd = no. of directions,
G_j, g_j = maximal and minimal gabarits for $j = 1, \ldots, nd$,
R_m^j= elementary coefficients; $j = 1, \ldots, nd$ and $m = 1, \ldots, ns$.

The both parameters of each elementary source could be associeted to the complex parameter

$$z_m = x_m + iy_m$$

and we denote

$$z = (z_1, \dots, z_m, \dots, z_{ns}) \in \mathbf{C}^{ns} \setminus \{0\}.$$

Let us consider the applications:

$$F_j(z) = \frac{1}{\sqrt{\sum_{l=1}^{ns} |z_l|^2}} \sum_{m=1}^{ns} z_m R_m^j, \quad f_j = 20 \log_{10} |F_j(z)|.$$

We set the below problem:

Find $z \in \mathbf{C}^{ns} \setminus \{0\}$ such that

$$g_j - f_j(z) \leq 0, \quad f_j(z) - G_j \leq 0, \quad \forall j = 1, \dots, nd. \tag{1}$$

In this paper, we shall present three approaches in order to solve the above problem: Unconstrained Optimization, Augmented Lagrangian Method and Nondifferentiable Optimization.

2 Unconstrained Optimization

Let us consider the cost function

$$J(z) = \frac{1}{2} \sum_{j=1}^{nd} \left[(\max\{0, g_j - f_j(z)\})^2 + (\max\{0, f_j(z) - G_j\})^2 \right]$$

and the unconstrained optimization problem

$$\left\{ \begin{array}{c} \inf J(z) \\ z \in \mathbf{C}^{ns} \setminus \{0\}. \end{array} \right. \tag{2}$$

This Least Squares type problem is introduced to solve the system (1).

Proposition 1 *The unconstrained optimization problem (2) has at least one solution.*

Proof. We use the change-of-variable

$$Z_m = z_m / \sqrt{\sum_{l=1}^{ns} |z_l|^2}.$$

The new problem is to minimize a continuous function over a compact set (the unit circle of $\mathbf{C}^{ns}$). □

Proposition 2 *For all z in $\mathbf{C}^{ns} \setminus \{0\}$, we have the equalities:*

$$\frac{\partial J}{\partial x_m}(z) = \left(-\max\{0, g_j - f_j(z)\} + \max\{0, f_j(z) - G_j\}\right) \frac{\partial f_j}{\partial x_m}(z),$$
$$\frac{\partial J}{\partial y_m}(z) = \left(-\max\{0, g_j - f_j(z)\} + \max\{0, f_j(z) - G_j\}\right) \frac{\partial f_j}{\partial y_m}(z),$$
$$\frac{\partial f_j}{\partial x_m}(z) = \frac{20}{\ln 10} \frac{1}{|F_j(z)|} \frac{\partial |F_j(z)|}{\partial x_m},$$
$$\frac{\partial f_j}{\partial y_m}(z) = \frac{20}{\ln 10} \frac{1}{|F_j(z)|} \frac{\partial |F_j(z)|}{\partial y_m},$$

$$\begin{aligned}\frac{\partial |F_j(z)|}{\partial x_m} &= \frac{1}{2|F_j(z)|} \frac{R_m^j \left(\sum_{l=1}^{ns} \overline{z}_l \overline{R}_l^j\right) + \left(\sum_{l=1}^{ns} z_l R_l^j\right) \overline{R}_m^j}{\sum_{l=1}^{ns} |z_l|^2} \\ &\quad - \frac{1}{2|F_j(z)|} \frac{\left(\sum_{l=1}^{ns} z_l R_l^j\right)\left(\sum_{l=1}^{ns} \overline{z}_l \overline{R}_l^j\right) 2x_m}{\left(\sum_{l=1}^{ns} |z_l|^2\right)^2},\end{aligned}$$

$$\begin{aligned}\frac{\partial |F_j(z)|}{\partial y_m} &= \frac{1}{2|F_j(z)|} \frac{(iR_m^j)\left(\sum_{l=1}^{ns} \overline{z}_l \overline{R}_l^j\right) - \left(\sum_{l=1}^{ns} z_l R_l^j\right)\left(i\overline{R}_m^j\right)}{\sum_{l=1}^{ns} |z_l|^2} \\ &\quad - \frac{1}{2|F_j(z)|} \frac{\left(\sum_{l=1}^{ns} z_l R_l^j\right)\left(\sum_{l=1}^{ns} \overline{z}_l \overline{R}_l^j\right) 2y_m}{\left(\sum_{l=1}^{ns} |z_l|^2\right)^2}.\end{aligned}$$

These formulas permit us to solve numerically the unconstrained optimization problem by the gradient type algorithms, where the gradient is computed analytically..

3 Augmented Lagrangian Method

We set $\mathcal{L} : \mathbf{C}^{ns} \setminus \{0\} \times \left(\mathbf{R}_+^{nd} \times \mathbf{R}_+^{nd}\right) \to \mathbf{R}$ as follow

$$\begin{aligned}\mathcal{L}(z; (\lambda, \mu)) &= \sum_{j=1}^{nd} \left[\lambda_j (g_j - f_j(z)) + \mu_j (f_j(z) - G_j)\right] \\ &\quad + \frac{r}{2} \sum_{j=1}^{nd} \left[(\max\{0, g_j - f_j(z)\})^2 + (\max\{0, f_j(z) - G_j\})^2\right],\end{aligned}$$

where the real number $r \geq 0$ is the penalization parameter.

We consider the problem:

Find a saddle point $\left(\overline{z};\left(\overline{\lambda},\overline{\mu}\right)\right)\in\mathbf{C}^{ns}\backslash\{0\}\times\left(\mathbf{R}_{+}^{nd}\times\mathbf{R}_{+}^{nd}\right)$ such that for all $z\in\mathbf{C}^{ns}\backslash\{0\}$ and all $(\lambda,\mu)\in\mathbf{R}_{+}^{nd}\times\mathbf{R}_{+}^{nd}$ we have

$$\mathcal{L}\left(\overline{z};(\lambda,\mu)\right)\leq\mathcal{L}\left(\overline{z};\left(\overline{\lambda},\overline{\mu}\right)\right)\leq\mathcal{L}\left(z;\left(\overline{\lambda},\overline{\mu}\right)\right). \tag{3}$$

The above problem is the second approach presented in this paper for solving the initial system (1).

The existence of a saddle point is an open problem. In order to solve numerically, we propose the following algorithm:

Uzawa like Algorithm

Step 1 Initialization: $r\geq 0,\ k=0,\ \lambda^{k}\in\mathbf{R}_{+}^{nd},\ \mu^{k}\in\mathbf{R}_{+}^{nd},\ \rho_{dual}\geq 0.$

Step 2 Compute z^k as $z^{k}\in\arg\min_{z\in\mathbf{C}^{ns}\backslash\{0\}}\mathcal{L}\left(z;\left(\lambda^{k},\mu^{k}\right)\right).$

Step 3 Stopping criterium:

If $\forall j\in\{1,\ldots,nd\},\ \left(g_j-f_j\left(z^k\right)\leq 0\right)\wedge\left(f_j\left(z^k\right)-G_j\leq 0\right)$ then z^k solution, STOP

Step 4 Update of λ^k and μ^k

$$\begin{aligned}\lambda_j^{k+1}&=\max\left\{0,\ \lambda_j^k+\rho_{dual}\left(g_j-f_j\left(z^k\right)\leq 0\right)\right\},\quad\forall j\in\{1,\ldots,nd\}\\ \mu_j^{k+1}&=\max\left\{0,\ \mu_j^k+\rho_{dual}\left(f_j\left(z^k\right)-G_j\leq 0\right)\right\},\quad\forall j\in\{1,\ldots,nd\}\\ k&=k+1\quad\text{Go to the Step 2}\end{aligned}$$

This algorithm combined with a multigrid algorithm, they have given very satisfactory numerical results (see [3]).

4 Nondifferentiable Optimization

For each $z\in\mathbf{C}^{ns}\backslash\{0\}$, we define

$$\psi(z)=\max_{j\in\{1,\ldots,nd\}}\max\left\{g_j-f_j(z),f_j(z)-G_j\right\}.$$

The function ψ is nonsmooth!

We consider the MinMax problem:

Find

$$z^{*}\in\arg\min_{z\in\mathbf{C}^{ns}\backslash\{0\}}\psi(z). \tag{4}$$

The above problem is the third approach presented in this paper for solving the initial system (1).

We have

$$\psi(z^*) \leq 0 \Leftrightarrow \forall j \in \{1, \ldots, nd\}, g_j - f_j(z^*) \leq 0, \quad f_j(z^*) - G_j \leq 0.$$

4.1 Existence

PROPOSITION 3 *The function $\psi : \mathbf{C}^{ns} \backslash \{0\} \to \mathbf{R}$ defined above is locally Lipschitz continuous.*

Proof. It is easy to check (see [4]). □

PROPOSITION 4 *The nonsmooth optimization problem has at least one solution.*

Proof. We use the change-of-variable $Z_m = z_m / \sqrt{\sum_{l=1}^{ns} |z_l|^2}$ The new problem is to minimize a continuous function over a compact set (the unit circle of $\mathbf{C}^{ns}$). □

4.2 Generalized Derivatives and Gradients

DEFINITION 1 *We say that the function $\psi : \mathbf{C}^{ns} \backslash \{0\} \to \mathbf{R}$ has a Gâteaux semiderivative at $z \in \mathbf{C}^{ns} \backslash \{0\}$ in the direction $h \in \mathbf{C}^{ns}$, if the following limit exists*

$$\lim_{t \searrow 0} \frac{\psi(z + th) - \psi(z)}{t}$$

Whenever it exists, it will be denoted by $d\psi(z; h)$. If $d\psi(x; h)$ exists for all $h \in \mathbf{C}^{ns}$, we say that ψ is Gâteaux semidifferentiable at z.

DEFINITION 2 *We say that the function $\psi : \mathbf{C}^{ns} \backslash \{0\} \to \mathbf{R}$ has a Clarke generalized directional semiderivative at $z \in \mathbf{C}^{ns} \backslash \{0\}$ in the direction $h \in \mathbf{C}^{ns}$, if the following limit exists*

$$\limsup_{\substack{t \searrow 0 \\ y \to z}} \frac{\psi(y + th) - \psi(y)}{t}$$

Whenever it exists, it will be denoted by $d_0\psi(z; h)$.

We introduce the notion of generalized gradient, following [1].

DEFINITION 3 *The Clarke's generalized gradient is defined as follows*

$$\partial\psi(z) = \{\xi \in \mathbf{C}^{ns};\, d_0\psi(z; h) \geq Re\, \langle \xi, h \rangle, \quad \forall h \in \mathbf{C}^{ns}\}.$$

PROPOSITION 5 (CLARKE) *The Clarke generalized gradient $\partial\psi(z)$ is nonempty, convex and compact subset.*

4.3 First order optimal conditions

PROPOSITION 6 (CLARKE) $d_0\psi(z,h) = \max_{\xi\in\partial\psi(z)} Re\,\langle \xi, h\rangle$.

PROPOSITION 7 (POLAK) *For all* $z \in \mathbf{C}^{ns}\setminus\{0\}$, $h \in \mathbf{C}^{ns}$, *there exits* $d\psi(z,h)$ *and* $d\psi(z,h) = d_0\psi(z,h)$.

If $\widehat{z} \in \mathbf{C}^{ns}\setminus\{0\}$ *is a local minimizer for* ψ, *then*

$$d\psi(\widehat{z},h) \geq 0, \quad \forall h \in \mathbf{C}^{ns}.$$

If $d\psi(z,h) < 0$ then h is a descent direction!

4.4 Steepest Descent Method for Nondifferentiable Problem

The descent direction h^k chosen by the Steepest Descent Method is

$$h^k = \arg\min_{\|h\|\leq 1} d\psi\left(z^k, h\right).$$

Before to present the algorithm, let us mention a simple result.

PROPOSITION 8 *We have*

$$\arg\min_{\|h\|\leq 1} d\psi\left(z^k,h\right) = \lambda \arg\min_{h\in\mathbf{C}^{ns}} \left\{ d\psi\left(z^k,h\right) + \frac{1}{2}\|h\|^2 \right\}$$

where $\lambda > 0$.

If we use the above proposition for computing the search direction, the Steepest Descent Method has the form:

Algorithm 1

Step 0 Initialization: $z^0 \in \mathbf{C}^{ns}\setminus\{0\}, \quad k = 0$

Step 1 Compute the search direction

$$h^k = \arg\min_{h\in\mathbf{C}^{ns}} \left\{ d\psi\left(z^k,h\right) + \frac{1}{2}\|h\|^2 \right\}$$

Step 2 Stopping criterium

If $d\psi\left(z^k,h^k\right) + \frac{1}{2}\left\|h^k\right\|^2 = 0$ then z^k is solution; STOP

Step 3 Compute the step size

$$\rho_k \in \arg\min_{\rho\geq 0} \psi\left(z^k + \rho h^k\right)$$

Step 4 Update: $z^{k+1} = z^k + \rho_k h^k$; $k = k+1$, Go to the Step 1

The Step 1 represents the principal difficulty.. In the next subsection, it's shown how to compute the search direction in the case when ψ is a Max function.

4.5 Steepest Descent Method for MinMax Problem

We begin with the following result:

PROPOSITION 9 (POLAK [4])

$$\partial\psi\left(z^k\right) = \Big\{\xi;\ \xi = -\sum_{j\in I_1\left(z^k\right)} \lambda_j \nabla f_j\left(z^k\right) + \sum_{j\in I_2\left(z^k\right)} \mu_j \nabla f_j\left(z^k\right)$$
$$\lambda_j \geq 0,\ \mu_j \geq 0,\ \sum_{j\in I_1\left(z^k\right)} \lambda_j + \sum_{j\in I_2\left(z^k\right)} \mu_j = 1\Big\}$$

where

$$I_1(z^k) = \{j;\ g_j - f_j\left(z^k\right) = \psi\left(z^k\right)\},\ \ I_2(z^k) = \Big\{j;\ f_j\left(z^k\right) - G_j = \psi\left(z^k\right)\Big\}.$$

Using the above result in the Algorithm 1, we obtain:

Algorithm 2

Step 0 Initialization: $z^0 \in \mathbf{C}^{ns}\setminus\{0\}$, $k = 0$, $\alpha, \beta \in (0,1)$

Step 1 Compute:

$$\theta\left(z^k\right) = -\min_{\lambda_j,\mu_j\geq 0, \sum_{j=1}^{nd}(\lambda_j+\mu_j)=1} \frac{1}{2}\left\|\sum_{j=1}^{nd}\left(-\lambda_j+\mu_j\right)\nabla f_j\left(z^k\right)\right\|^2$$
$$+\sum_{j=1}^{nd}\Big[\lambda_j\left(\psi\left(z^k\right) - g_j + f_j\left(z^k\right)\right) + \mu_j\left(\psi\left(z^k\right) - f_j\left(z^k\right) + G_j\right)\Big]$$

$$(\lambda_j, \mu_j) \in \arg\min_{\lambda_j,\mu_j\geq 0, \sum_{j=1}^{nd}(\lambda_j+\mu_j)=1} \{\ldots\}$$

Step 2 Stopping criterium: If $\theta\left(z^k\right) = 0$ then z^k is solution; STOP else $h^k = \sum_{j=1}^{nd}\left(\lambda_j - \mu_j\right)\nabla f_j\left(z^k\right)$

Step 3 Compute the step size (Armijo's rule)

$$\rho_k = \max\Big\{\rho;\ \rho = \beta^p;\ p \in \mathbf{N},\ \psi\left(z^k + \rho h^k\right) - \psi\left(z^k\right) \leq \alpha\rho\theta\left(z^k\right)\Big\}$$

Step 4 Update: $z^{k+1} = z^k + \rho_k h^k;\ k = k+1,$ Go to the Step 1

At the Step 1, we have to solve a quadratic problem in order to get the descent direction. This quadratic problem could be solved by the classical algorithm of Frank and Wolfe.

The Frank-Wolfe Algorithm for Quadratic Problems

$$\inf f(x) = \frac{1}{2}x^T Q x + c^T x, \quad x \in P = \left\{ x \in \mathbf{R};\, x_i \geq 0,\, \sum_{i=1}^{n} x_i = 1 \right\}$$

where Q is a symmetric and semi-positive defined matrix.

Step 0 Find $x^0 \in P$ using two phases simplex method. Set $k = 0$

Step 1 If $\nabla f\left(x^k\right) = 0$ then x^k is a solution; Stop.

Step 2 Find a solution $\overline{x}^k$ of the linear optimization problem

$$\inf x^T \nabla f\left(x^k\right), \quad x \in P$$

Step 3 If $\left(\overline{x}^k - x^k\right)^T \nabla f\left(x^k\right) = 0$ then x^k is a solution; Stop.

Step 4 If $\left(\overline{x}^k - x^k\right)^T \nabla f\left(\overline{x}^k\right) \leq 0$ then set $\lambda_k = 1$ else set

$$\lambda_k = -\frac{\left(\overline{x}^k - x^k\right)^T \nabla f\left(x^k\right)}{\left(\overline{x}^k - x^k\right)^T Q \left(\overline{x}^k - x^k\right)} \in (0,1)$$

Step 5 Update: $x^{k+1} = x^k + \lambda_k \left(\overline{x}^k - x^k\right);\ k = k+1$, Go to the step 1

An enhanced version of the Frank-Wolfe algorithm is described in [2]. In the case $G_j - g_j =$ constant, which is frequently in practice, the Step 1 can be rewritten and we obtain the below algorithm:

Algorithm 3 (in the case $G_j - g_j =$ constant)

Step 0 Initialization: $z^0 \in \mathbf{C}^{ns} \setminus \{0\}, \quad k = 0, \quad \alpha, \beta \in (0,1)$

Step 1 Compute:

$$\theta\left(z^k\right) = -\min_{\sum_{j=1}^{nd} |\gamma_j| = 1} \frac{1}{2} \left\| \sum_{j=1}^{nd} \gamma_j \nabla f_j \left(z^k\right) \right\|^2 + \sum_{j=1}^{nd} \gamma_j \left(\frac{G_j + g_j}{2} - f_j\left(z^k\right) \right) + \psi\left(z^k\right) + \frac{G_j - g_j}{2}$$

$$\gamma^k = \left(\gamma_j^k\right)_{1\le j\le nd} \in \arg \min_{\sum_{j=1}^{nd}|\gamma_j|=1} \{\ldots\}$$

Step 2 Stopping criterium: If $\theta\left(z^k\right) = 0$ then z^k is solution; STOP else $h^k = -\sum_{j=1}^{nd} \gamma_j^k \nabla f_j\left(z^k\right)$

Step 3 Compute the step size (Armijo's rule)

$$\rho_k = \max\left\{\rho;\ \rho = \beta^p;\ p \in \mathbf{N},\ \psi\left(z^k + \rho h^k\right) - \psi\left(z^k\right) \le \alpha\rho\theta\left(z^k\right)\right\}$$

Step 4 Update: $z^{k+1} = z^k + \rho_k h^k;\ k = k+1,$ Go to the Step 1

We observe that at the Step 1, the dimension of the problem has reduced at half, but the constraint is more difficult.

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Anexa

In this subsection, we show the numerical results obtained using the Algorithm 2 combined with Frank-Wolfe algorithm for a problem with $ns = 64$ and $nd = 240$.

The minimizing function is ψ and the θ is defined at the Step 1 of the Algorithm 2.

Step	ψ	θ
0	48.2213	-5823.72
1	43.0761	-81.3748
2	33.4057	-30.7252
3	29.3392	-22.216
4	27.5251	-0.150219
5	27.2241	-66.2132
6	41.3246	-80.4636
7	27.3999	-66.4605
8	27.4219	-66.7833
9	27.4866	-66.9601
10	27.5495	-0.091815
11	27.3619	-0.0928569
12	27.2093	-66.2294
13	27.2818	-0.0644296
14	27.1928	-0.0142176

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