Domain decomposition method for a flow through two porous media

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Abstract. A flow through two porous media is studied. The two media are in contact. The domain decomposition is obtained by introducing a Lagrange multiplier. The aim of this paper is to present a new proof for the existence and uniqueness of the Lagrange multiplier. This technique can be used for time-dependent partial differential equations where the other methods, like convex optimization, differential optimization, hybrid method, fail.

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1 Introduction

We study the flow through two porous media Ω_1 and Ω_2. The two media are in contact, it means that $\partial\Omega_1 \cap \partial\Omega_2 = \Gamma$, as in the Figure 1.

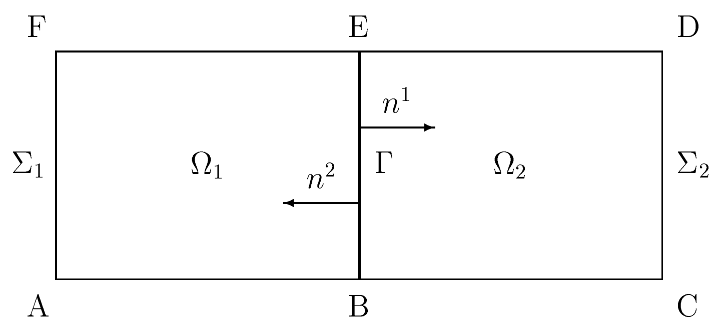

Figure 1: The porous media

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We denote by Σ_1 and Σ_2 the faces $]F, A[$ and $]C, D[$ respectively.

The porosity in Ω_1 and Ω_2 is given by the applications $k_1 : \overline{\Omega_1} \to \mathbb{R}$ and $k_2 : \overline{\Omega_2} \to \mathbb{R}$ respectively. We assume that

$$\begin{aligned} \exists \alpha_1 &> 0, \forall x \in \overline{\Omega_1},\ k_1(x) \geq \alpha_1, \\ \exists \alpha_2 &> 0, \forall x \in \overline{\Omega_2},\ k_2(x) \geq \alpha_2. \end{aligned}$$

Let us consider $g_1 : \Sigma_1 \to \mathbb{R}$ and $g_2 : \Sigma_2 \to \mathbb{R}$.

The classical equations for the flow through the porous media Ω_1 and Ω_2 are the following

$$-\text{div}\,(k_1(x)\nabla u_1(x)) = 0 \text{ on } \Omega_1 \tag{1}$$

$$-\text{div}\,(k_2(x)\nabla u_2(x)) = 0 \text{ on } \Omega_2 \tag{2}$$

$$u_1(x) = g_1(x) \text{ on } \Sigma_1 \tag{3}$$

$$u_2(x) = g_2(x) \text{ on } \Sigma_2 \tag{4}$$

$$k_1(x)\nabla u_1(x).n^1(x) = 0 \text{ on }]AB[\cup]EF[\tag{5}$$

$$k_2(x)\nabla u_2(x).n^2(x) = 0 \text{ on }]DE[\cup]BC[\tag{6}$$

$$k_1(x)\nabla u_1(x).n^1(x) = k_2(x)\nabla u_2(x).n^2(x) \text{ on } \Gamma \tag{7}$$

$$u_1(x) = u_2(x) \text{ on } \Gamma \tag{8}$$

where n^1 is the unit outward normal to $\partial\Omega_1$ and n^2 is the unit outward normal to $\partial\Omega_2$.

The equalities (7) and (8) represent the contact boundary conditions.

If the function λ is known, where

$$\lambda(x) = k_1(x)\nabla u_1(x).n^1(x) = k_2(x)\nabla u_2(x).n^2(x) \text{ on } \Gamma,$$

we can solve the equation (1) with the boundary conditions (3), (5) and

$$k_1(x)\nabla u_1(x).n^1(x) = \lambda(x) \text{ on } \Gamma.$$

Also, we can solve the equation (2) with the boundary conditions (4), (6) and

$$k_2(x)\nabla u_2(x).n^2(x) = \lambda(x) \text{ on } \Gamma.$$

We observe that the equations in Ω_1 and Ω_2 can be solved independently, so we can solve numerically using parallel computation. But the function λ is not known a priori.

We can split the system of equations by introducing a Lagrange multiplier.

The aim of this paper is to present a new proof for the existence of a Lagrange multiplier. This kind of proof can be used in the case of time-dependent equations, where the other methods fail.

2 Variational formulation

We assume that Ω_1 and Ω_2 are two bounded Lipschitz domains. Let g_1 and g_2 be given in $H^{1/2}(\Sigma_1)$ and $H^{1/2}(\Sigma_2)$ respectively. There exist $\overline{u}_1$ in $H^1(\Omega_1)$ and $\overline{u}_2$ in $H^1(\Omega_2)$, such that $\overline{u}_1 = g_1$ on Σ_1, $\overline{u}_2 = g_2$ on Σ_2 and $\overline{u}_1 = \overline{u}_2 = 0$ on Γ.

Let k_1 and k_2 be given in $L^\infty(\Omega_1)$ and $L^\infty(\Omega_2)$ respectively. We define the bilinear form a_1 from $H^1(\Omega_1) \times H^1(\Omega_1)$ to $\mathbb{R}$ by

$$a_1(u_1, v_1) = \int_{\Omega_1} k_1(x) \nabla u_1(x) . \nabla v_1(x)\, dx$$

and the bilinear form a_2 from $H^1(\Omega_2) \times H^1(\Omega_2)$ to $\mathbb{R}$ by

$$a_2(u_2, v_2) = \int_{\Omega_1} k_2(x) \nabla u_2(x) . \nabla v_2(x)\, dx.$$

We denote

$$\begin{aligned} V_1 &= \left\{ v_1 \in H^1(\Omega_1);\ v_1 = 0 \text{ on } \Sigma_1 \right\}, \\ V_2 &= \left\{ v_2 \in H^1(\Omega_2);\ v_2 = 0 \text{ on } \Sigma_2 \right\}, \\ V &= \{ V_1 \times V_2;\ v_1 = v_2 \text{ on } \Gamma \}. \end{aligned}$$

The variational formulation for the system (1)–(8) is the following:
find $(u_1 - \overline{u}_1, u_2 - \overline{u}_2)$ in V, such that

$$a_1(u_1 - \overline{u}_1, v_1) + a_2(u_2 - \overline{u}_2, v_2) = -a_1(\overline{u}_1, v_1) - a_2(\overline{u}_2, v_2), \quad \forall (v_1, v_2) \in V. \tag{9}$$

As a consequence of the Lax-Milgram Theorem, there exists a unique solution for the above variational system.

3 Some known ways to prove the existence of the Lagrange multiplier

Let us denote by $\gamma_\Gamma^1 : H^1(\Omega_1) \to H^{1/2}(\Gamma)$ and $\gamma_\Gamma^2 : H^1(\Omega_2) \to H^{1/2}(\Gamma)$ the trace applications. Also, we denote $M = H^{1/2}(\Gamma)$.

We consider the linear operator

$$\begin{gathered} B : V_1 \times V_2 \to M, \\ B(v_1, v_2) = \gamma_\Gamma^1(v_1) - \gamma_\Gamma^2(v_2). \end{gathered}$$

3.1 Convex optimization

Since a_1 and a_2 are symmetric forms, the variational system (9) is equivalent to convex optimization below.
Find $(u_1 - \overline{u}_1, u_2 - \overline{u}_2)$ in $V_1 \times V_2$, an optimal solution for

$$\inf_{(v_1,v_2)\in V_1\times V_2} J(v_1, v_2) \tag{10}$$

subject to

$$B(v_1, v_2) = 0 \tag{11}$$

where

$$J(v_1, v_2) = \frac{1}{2}a_1(v_1, v_1) + \frac{1}{2}a_2(v_2, v_2) + a_1(\overline{u}_1, v_1) + a_2(\overline{u}_2, v_2)$$

Let us consider the set

$$\begin{aligned} C \;=\; & \left\{(\alpha, z) \in \mathbb{R}_+^* \times M;\, \exists\,(v_1, v_2) \in V_1 \times V_2, \right. \\ & \left. 0 \le J(v_1, v_2) - J(u_1 - \overline{u}_1, u_2 - \overline{u}_2) + \alpha,\; B(v_1, v_2) = z\right\} \end{aligned}$$

which is convex. In view of a separation theorem for convex sets, there exits a hyperplane which separates C from $\{(0, 0)\}$. Since the following interior regularity of the constraints holds

$$0 \in \operatorname{int} B(V_1 \times V_2), \tag{12}$$

it follows the existence of the Lagrange multiplier λ in M', where M' is the dual space of M, such that

$$\begin{gathered} \forall\,(v_1, v_2) \in V_1 \times V_2, \\ a_1(u_1, v_1) + a_2(u_2, v_2) + \langle \lambda, B(v_1, v_2)\rangle_{M',M} = 0 \end{gathered} \tag{13}$$

or equivalent

$$a_1(u_1 - \overline{u}_1, v_1) = -a_1(\overline{u}_1, v_1) - \left\langle \lambda, \gamma_\Gamma^1(v_1)\right\rangle_{M',M} \quad \forall v_1 \in V_1, \tag{14}$$

$$a_2(u_2 - \overline{u}_2, v_2) = -a_2(\overline{u}_2, v_2) + \left\langle \lambda, \gamma_\Gamma^2(v_2)\right\rangle_{M',M} \quad \forall v_2 \in V_2 \tag{15}$$

where $\langle \cdot, \cdot \rangle_{M',M}$ is the duality product between M' and M.

The regularity condition (12) holds because the operator B is surjective.

The variational equations (14) and (15) can be solved numerically in parallel..

3.2 Differentiability optimization

The existence of a Lagrange multiplier can be proved without convex assumptions, but under differentiability hypothesis.

We recall the Theorem 1.13 from the book [2, p. 194].

Theorem 1 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two Banach spaces. Let $f : X \to Y$ be a Fréchet differentiable function in $x_0 \in X$ and $T \in \mathcal{L}(X, Y)$ with closed range. If x_0 is an optimal solution of the problem*

$$\inf \{f(x); T(x) = k\}$$

where k is in Y, then there exits $y_0^* \in Y'$ such that

$$f'_{x_0}(x) + \langle y_0^*, T(x) \rangle_{Y',Y} = 0, \quad \forall x \in X.$$

Using this result in the case $X = V_1 \times V_2$, $Y = M$, $f = J$, $T = B$, $k = 0$, $x_0 = (u_1, u_2)$, $y_0^* = \lambda$, we obtain the existence of the Lagrange multiplier λ, such that the variational equations (14) and (15) hold.

3.3 Hybrid formulation

The results concerning the existence of the Lagrange multiplier, which are more popular in the Finite Element community, are due to Babuska [1] and Brezzi [4].

We recall the principal result.

Theorem 2 *Let W and M be two Hilbert spaces. Let us consider two bilinear and continuous forms $a : W \times W \to \mathbb{R}$ and $b : W \times M' \to \mathbb{R}$, such that*

$$\exists \alpha > 0, \forall v \in V, \quad a(v, v) \geq \alpha \|v\|^2 \tag{16}$$

$$\exists \beta > 0, \quad \inf_{\|\mu\|=1} \sup_{\|w\|=1} b(w, \mu) \geq \beta. \tag{17}$$

where

$$V = \{v \in W;\, b(v, \mu) = 0, \forall \mu \in M'\}.$$

Then for each f in W', there exists an unique solution $(u, \lambda) \in W \times M'$ such that

$$\begin{cases} a(u, w) + b(w, \lambda) = \langle f, w \rangle_{W',W} & \forall w \in W, \\ b(u, \mu) = 0, & \forall \mu \in M'. \end{cases} \tag{18}$$

We shall use this result for in the case $W = V_1 \times V_2$, $M = H^{1/2}(\Gamma)$,

$$\begin{aligned} a((u_1, u_2), (v_1, v_2)) &= a_1(u_1, v_1) + a_2(u_2, v_2), \\ b((v_1, v_2), \mu) &= \langle \mu, B(v_1, v_2) \rangle_{M',M}. \end{aligned}$$

The relation (16) holds because a_1 is V_1 elliptic and a_2 is V_2 elliptic..

In view of the fact that B is surjective and using the characterization theorem for the surjective operators [3, p. 29], we obtain that the *inf-sup* condition (17) holds.

The first equation of (18) is equivalent to (13).

4 New proof for the existence and uniqueness of the Lagrange multiplier

Theorem 3 *Let (u_1, u_2) be the solution of the variational problem (9). Then there exists an unique element λ of M', such that the relation (13) holds.*

Proof. **Uniqueness.** We suppose that there exists two Lagrange multiplier λ_1 and λ_2. From the equality (13), we obtain that

$$\langle \lambda_1 - \lambda_2, B(v_1, v_2) \rangle_{M',M} = 0, \quad \forall (v_1, v_2) \in V_1 \times V_2.$$

But the operator B is surjective because $\overline{\Sigma}_1 \cap \overline{\Gamma} = \emptyset$ and $\overline{\Sigma}_2 \cap \overline{\Gamma} = \emptyset$. Consequently, we obtain that $\lambda_1 - \lambda_2 = 0$.

Existence. Let g be in M. Since $\overline{\Sigma}_1 \cap \overline{\Gamma} = \emptyset$, we have that the application trace γ_Γ^1 is surjective and it follows that there exists v_1 in V_1, such that $\gamma_\Gamma^1(v_1) = g$. Analogous, there exists v_2 in V_2, such that $\gamma_\Gamma^2(v_2) = g$.

We define the Lagrange multiplier as follows:

$$\lambda(g) \overset{def}{=} a_1(u_1, v_1).$$

We must to prove that λ is well defined. For other $\widetilde{v}_1$ in V_1, such that $\gamma_\Gamma^1(\widetilde{v}_1) = g$, using the equality (9), we obtain that

$$a_1(u_1, \widetilde{v}_1) = -a_2(u_2, v_2) = a_1(u_1, v_1).$$

Then λ is well defined and we have that

$$\begin{aligned} a_1(u_1, v_1) &= \lambda\left(\gamma_\Gamma^1(v_1)\right), \quad \forall v_1 \in V_1, \\ a_2(u_2, v_2) &= -\lambda\left(\gamma_\Gamma^2(v_2)\right), \quad \forall v_2 \in V_2. \end{aligned} \tag{19}$$

From the continuity of the forms a_1 and a_2, there exist two constants C_1 and C_2 such that

$$\begin{aligned} |a_1(u_1, v_1)| &\leq C_1 \|v_1\|_{1,\Omega_1} \quad \forall v_1 \in V_1, \\ |a_2(u_2, v_2)| &\leq C_2 \|v_2\|_{1,\Omega_2} \quad \forall v_2 \in V_2 \end{aligned} \tag{20}$$

where $\|\cdot\|_{1,\Omega_1}$ and $\|\cdot\|_{1,\Omega_2}$ are the standard norms of the Sobolev spaces $H^1(\Omega_1)$ and $H^1(\Omega_2)$ respectively.

We prove now that λ is linear.

Let g and h be in M and α and β in $\mathbb{R}$. Then there exist v_1 and w_1 in V_1 such that $\gamma_\Gamma^1(v_1) = g$ and $\gamma_\Gamma^1(w_1) = h$ and we have

$$\gamma_\Gamma^1(\alpha v_1 + \beta w_1) = \alpha g + \beta h\,.$$

From the definition of λ and since a_1 is bilinear, we obtain

$$\begin{aligned}\lambda(\alpha g + \beta h) &= a_1(u_1, \alpha v_1 + \beta w_1)\\ &= \alpha a_1(u_1, v_1) + \beta a_1(u_1, w_1)\\ &= \alpha\lambda(g) + \beta\lambda(h)\,.\end{aligned}$$

We prove now that λ is continuous.

From the relations (19) and (20), we have

$$|\lambda(g)| \leq C_1 \|v_1\|_{1,\Omega_1} \quad \forall v_1 \in V_1,\ \gamma_\Gamma^1(v_1) = g.$$

It follows that

$$|\lambda(g)| \leq C_1 \ \inf\left\{\|v_1\|_{1,\Omega_1};\ \gamma_\Gamma^1(v_1) = g\right\}.$$

Since the application γ_Γ^1 is surjective, as a consequence of the Banach theorem (see [3, p. 19]), we obtain that the norm

$$\|g\|_* \stackrel{def}{=} \inf\left\{\|v_1\|_{1,\Omega_1};\ \gamma_\Gamma^1(v_1) = g\right\}$$

is equivalent to the norm $\|\cdot\|_{1/2,\Gamma}$ of the Sobolev space $H^{1/2}(\Gamma)$. Consequently, λ is continuous, which completes the proof. $\square$

5 Conclusions

We have presented a new proof for the existence and uniqueness of the Lagrange multiplier for a contact problem. The variational system can by decomposed by introducing a Lagrange multiplier. The both variational systems obtained after the decomposition could be solved numerically by parallel computing.

This kind of proof can be used for time-dependent partial differential equations (see [5]) where the other methods, like convex optimization, differential optimization, hybrid method, fail.

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