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# A stable algorithm for a fluid-structure interaction problem in 3D

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## Abstract

A fluid-structure interaction problem is studied in the following hypotheses: the fluid is incompressible and it is governed by the time-dependent Stokes equations and the sructure is governed by the linear elasticity equations.

The interaction beetwen the blood and the left ventricle of the heart is the physical system which is modeled.

This paper presents a stable algorithm wich computs the velocity and the pressure of the fluid and the velocity of the structure, using the Lagrange multiplier method.

## 1 Introduction

A mathematical model for the biomechanics contact problems beetwen the blood and an elastic structure has been studied by J.L. Lions in [5, p. 120–129] and [3, vol. 8, chap. XVIII, p. 795–801]. In the linear case, i.e. the fluid is incompressible and it is governed by the time-dependent Stokes equations and the sructure is governed by linear elasticity equations, it has been proved the existence and the uniqueness of the solution to 3D problem.

In order to approximate the solution of this model, we propose an algorithm where the Lagrange multipliers are used to treat the both constraints of the problem: the free-divergence and the continuity of the velocity on the contact surface.

We can decouple the problem by introducing the Lagrange multiplier to treat the boundary value on the contact surface. This fact has positive consequences: we can use the existent theories and the numerical procedures to solve the fluid problem and the structure problem, separately.

Another attractive point is the simplicity of the implementation: at each time step, the linear system is solved by Uzawa algorithm. At each iteration, the algorithm solves two decouplated problems, one for the fluid and one for the structure, the both problems have as parameter the Lagrange multiplier on the contact surface. The Uzawa algorithm finds the contact Lagrange multiplier, such that the velocities of fluid and structure are equal on the contact surface.

In this paper, a discret time algorithm is presented.

In the second section, it is proved that the discret time problem is well defined, using the results of Babuska [1] and Brezzi [2]. Firstly, the surjectivity of the constraint operator is proved. Using a characterization theorem of the surjective operators, the *inf-sup* condition is obtained.

In the third section, the time stability of the algorithm is proved. The proof is originaly: we don't use separately the existent stability theories for the fluid problem, respectively for the structure problem, but we prove the stability for the fluid and for the structure, simultaneously. The velocity is evaluated in the norm $H^1$, hence better than the usual evaluation in the norm $L^2$ used in the approximation of Stokes equation.

## 2 Discret time variational formulation

Let $\Omega^F$ (resp. $\Omega^S$) be the domain in $\mathbf{R}^3$ of the fluid (resp. of the structure), such that: $\overline{\Omega^F} \cap \overline{\Omega^S} = \Gamma$, $\partial\Omega^F = \Gamma$ and $\partial\Omega^S = \overline{\Gamma} \cup \overline{\Sigma^1} \cup \overline{\Sigma^2}$.

Let $W^1 = H^1(\Omega^F)^N$ and $W^2 = \{w^2 \in H^1(\Omega^S)^N, w^2 = 0 \text{ on } \Sigma^1\}$ be the velocity spaces for the fluid and for the structure.

Let $a_F$ and $a_S$ be two bilinear applications, defined by:

$$\begin{cases} a_F : W^1 \times W^1 \longrightarrow \mathbf{R} \\ a_F(v, w^1) = \displaystyle\sum_{i,j=1}^{N} \int_{\Omega^F} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i^1}{\partial x_j} dx \end{cases}$$

$$\begin{cases} a_S : W^2 \times W^2 \longrightarrow \mathbf{R} \\ a_S(\nu, w^2) = \displaystyle\sum_{i,j=1}^{N} \int_{\Omega^S} \sigma_{ij}(\nu)\epsilon_{ij}(w^2) dx \end{cases}$$

where $N = 3$, $\sigma_{ij} = \lambda^S(\sum_{k=1}^{N} \epsilon_{kk}) + 2\mu^S\epsilon_{ij}$, $\epsilon_{ij}(w^2) = \frac{1}{2}(\frac{\partial w_i^2}{\partial x_j} + \frac{\partial w_j^2}{\partial x_i})$ and $\lambda^S,\ \mu^S > 0$.

The following quantities are given:

**i)** $f_1^{n+1} \in L^2(\Omega^F)^N$ and $f_2^{n+1} \in L^2(\Omega^S)^N$

**ii)** $\rho \in \mathbf{R}_+$ the density of the structure, $\nu_{cin} = \frac{\mu^F}{\rho^F} \in \mathbf{R}_+$ the cinematical viscosity

**iii)** $v^n \in W^1$ the velocity of the fluid

**iv)** $\nu^0, ..., \nu^n \in W^2$ the velocities of the structure, such that $v^0 = \nu^0$

**v)** $u_0 \in W^2$ the initial displacement of the structure

Our discret time problem has the following form:

*Find* $(v^{n+1}, \nu^{n+1}, p^{n+1}, \lambda^{n+1}) \in W^1 \times W^2 \times Q \times M$ *where* $Q = L^2(\Omega^F)$ *and* $M = \overline{H^{1/2}(\Gamma)}^N$, *such that:*

$$\frac{1}{\Delta t}(v^{n+1}, w^1)_{0,\Omega^F} + \nu_{cin} a_F(v^{n+1}, w^1) + \frac{1}{\Delta t}(\nu^{n+1}, w^2)_{0,\Omega^S} + \frac{\Delta t}{2\rho} a_S(\nu^{n+1}, w^2)$$
$$- (\mathbf{div}\ w^1, p^{n+1})_{0,\Omega^F} - \left(\gamma_0(w^1) - \gamma_\Gamma(w^2), \lambda^{n+1}\right)_{1/2,\Gamma}$$
$$= (f_1^{n+1}, w^1)_{0,\Omega^F} + (f_2^{n+1}, w^2)_{0,\Omega^S}$$
$$- \frac{1}{\rho} a_S\left(u_0 + \frac{\Delta t}{2}(\nu^0 + 2\sum_{i=1}^{n} \nu^i), w^2\right) + \frac{1}{\Delta t}(v^n, w^1)_{0,\Omega^F} + \frac{1}{\Delta t}(\nu^n, w^2)_{0,\Omega^S}$$
$$\forall w^1 \in W^1 \text{ and } \forall w^2 \in W^2 \tag{1}$$

$$(\mathbf{div}\ v^{n+1}, q)_{0,\Omega^F} = 0, \quad \forall q \in Q \tag{2}$$

$$\left(\gamma_0(v^{n+1}) - \gamma_\Gamma(\nu^{n+1}), \mu\right)_{1/2,\Gamma} = 0, \quad \forall \mu \in M \tag{3}$$

where $\gamma_0 : H^1(\Omega^F)^N \longrightarrow H^{1/2}(\Gamma)^N$ is the trace application and $\gamma_\Gamma : H^1(\Omega^S)^N \longrightarrow H^{1/2}(\Gamma)^N$ is the restriction on $\Gamma$ of the trace application.

**Remark 1** At each time step, we have to solve the system with Lagrange multipliers (1)-(3). This is a Babuska-Brezzi type variational problem, where the Lagrange multipliers are $p$ and $\lambda$, which treat respectively the two constraints of the problem: the free-divergence for the fluid and the continuity of the velocity on the contact interface.

The system (1)-(3) is the proposed algorithm which approximates the solution of the mathematical model studied by J.L. Lions. □

We shall show that the discret time problem is well defined using the results of Babuska and Brezzi. Firstly, the surjectivity of the constraints operator is proved.

**Proposition 1** *Let* $\Omega^F$ *and* $\Omega^S$ *be two open bounded sets in* $\mathbf{R}^3$ *and* $\overline{\Omega^F} \cap \overline{\Omega^S} = \overline{\Gamma}$, $\partial\Omega^S = \overline{\Gamma} \cup \overline{\Sigma^1} \cup \overline{\Sigma^2}$ *such that:*

$$\begin{cases} \Omega^F \in \mathcal{C}^{2,1} \text{ and } \Omega^S \in \mathcal{C}^{0,1} \\ mes(\Gamma) > 0,\ mes(\Sigma^1) > 0,\ mes(\Sigma^2) > 0 \\ \overline{\Gamma} \cap \overline{\Sigma^1} = \emptyset \\ \Gamma,\ \Sigma^1,\ \Sigma^2 \text{ are manifolds in } \mathbf{R}^{N-1} \text{ of } \mathcal{C}^2 \text{ class} \end{cases} \tag{4}$$

*Then the constraint operator* $B$ *defined by:*

$$\begin{cases} B : W^1 \times W^2 \longrightarrow Q \times M \\ B(w^1; w^2) = (\mathbf{div}\ w^1; \gamma_0(w^1) - \gamma_\Gamma(w^2)) \end{cases} \tag{5}$$

*is surjective and continuous.*

*Proof:* The proof is divided in two steps:

**1st Step:** It is proved that the operator **div** from $H^1(\Omega^F)^N$ into $L^2(\Omega^F)$ is surjective. The following classical result is used:

*If $\Omega^F \in \mathcal{C}^{2,1}$ and $q \in L^2(\Omega^F)$, then there exists an unique solution $u \in H^2(\Omega^F) \cap H^1_0(\Omega^F)$ to the system:*

$$\Delta u = q \tag{6}$$

Since $\Delta u = \mathbf{div}(\mathbf{grad}\ u)$ and if we set $w^1 = \mathbf{grad}\ u$, we obtain $w^1 \in H^1(\Omega^F)$.

**2nd Step**: As a consequence of the theorem (7.2) due to Kikuki & Oden [4, p. 177], we have the following result:

If $\Omega^S \in \mathcal{C}^{0,1}$, then for all $g \in H^{1/2}(\Gamma)$ we can find $w^2 \in H^1(\Omega^S)$ such that:

$$\begin{cases} \gamma_\Gamma(w^2) = g \\ \gamma_{\Sigma^1}(w^2) = 0 \end{cases} \tag{7}$$

Now, the surjectivity of $B$ can be proved. Let $(q, \mu) \in Q \times M$. According to the first step, there exists $w^1 \in W^1$ such that $\mathbf{div}\ w^1 = q$ and from the second step, there exists $w^2 \in W^2$ such that:

$$\gamma_\Gamma(w^2) = -\mu + \gamma_0(w^1) \tag{8}$$

therefore the surjectivity is obtained.

The continuity of $B$ is a simple consequence of the continuity of the operators trace and **div**. □

**Proposition 2** *Under the same hypotheses of the proposition 1, the condition* **inf-sup** *holds, i.e. there exists $\alpha > 0$ such that:*

$$\forall q \in Q, \forall \mu \in M, \quad \alpha(\| q \|_{0,\Omega^F} + \| \mu \|_{1/2,\Gamma})$$
$$\leq \sup_{\substack{(w^1,w^2)\in W^1\times W^2 \\ (w^1,w^2)\neq(0,0)}} \frac{|\ (\mathbf{div}\ w^1, q) + \Big(\gamma_0(w^1) - \gamma_\Gamma(w^2), \mu\Big)_{1/2,\Gamma}\ |}{(\| w^1 \|^2_{1,\Omega^F} + \| w^2 \|^2_{1,\Omega^S})^{1/2}}$$

The proposition 2 is a consequence of the closed range theorem and the proposition 1.

**Theorem 1** *Under the same hypotheses of the proposition 1, the discret time problem (1)-(3) has an unique solution.*

*Proof*: Let $X = W^1 \times W^2$. The application:

$$a : X \times X \longrightarrow \mathbf{R}, \text{ defined by}$$
$$a((v, \nu); (w^1, w^2))$$
$$= \frac{1}{\Delta t}(v, w^1)_{0,\Omega^F} + \nu_{cin} a_F(v, w^1) + \frac{1}{\Delta t}(\nu, w^2)_{0,\Omega^S} + \frac{\Delta t}{2\rho} a_S(\nu, w^2)$$

is $X-$elliptic, because $a_F$ is coercive, $a_S$ is $W^2-$elliptic and all the constants are positive. Now, we can use the theory of Babuska-Brezzi. In view of this theory and the proposition 2, the conclusion holds. □

## 3 Time stability of the algorithm

In this section the time stability of the algorithm defined by the equations (1)–(3) is proved, under certains asumptions. The proof is originaly: we don't use separately the existent stability theories for the fluid problem, respectively for the structure problem, but we prove the stability for the fluid and the structure, simultaneously. The velocity is evaluated in the norm $H^1$, hence better than the usual evaluation in the norm $L^2$ used in the approximation of Stokes equation.

**Hypothesis 1**

$$(f_1^n, f_2^n) = (f_1, f_2) \in H, \qquad \forall n \geq 0$$

**Hypothesis 2**
There exists $K_1$ a constant, which doesn't depend on $\Delta t$, such that:

$$\| \nu^1 \|_{1,\Omega^S} \leq K_1, \qquad \forall \Delta t \leq T$$

**Hypothesis 3**
There exists $K_2$ a constant, which doesn't depend on $\Delta t$, such that:

$$\left( \| v^1 - v^0 \|_{0,\Omega^F}^2 + \| \nu^1 - \nu^0 \|_{0,\Omega^S}^2 \right)^{1/2} \leq (\Delta t) K_2, \qquad \forall \Delta t \leq T$$

**Remark 2** In the hypotheses 2 and 3, only the initial data and the first step of the algorithm (1)-(3) are concerned. □

**Theorem 2** *Under the hypotheses 1, 2 and 3, the algorithm defined by the equations (1)–(3) is not-conditionaly stable in the following sens: there exists a constant $K$, which doesn't depend upon $n$ and $\Delta t$, such that:*

$$\| v^n \|_{1,\Omega^F} + \| \nu^n \|_{1,\Omega^S} + \| p^n \|_{0,\Omega^F} + \| \lambda^n \|_{1/2,\Gamma} \leq K, \qquad (9)$$

*for all $n$ and $\Delta t$ verifying $n(\Delta t) \leq T$.*

*Proof:* The demonstration is divided in tree steps:

1) $\exists K_3 > 0$ such that $\| v^n \|_{0,\Omega^F} + \| \nu^n \|_{0,\Omega^S} \leq K_3$
2) $\exists K_4 > 0$ such that $\| v^n \|_{1,\Omega^F} + \| \nu^n \|_{1,\Omega^S} \leq K_4$
3) $\exists K_5 > 0$ such that $\| p^n \|_{0,\Omega^F} + \| \lambda^n \|_{1/2,\Gamma} \leq K_5$

**1st Step:** We write the equality (1) for $n$, after we substract it from the relation (1) written for $n+1$. We substitute $w^1$ by $v^{n+1} - v^n$ and $w^2$ by $\nu^{n+1} - \nu^n$. From the equalities (2) and (3), the terms which contain $p^n$ and

$\lambda^n$ will disappear. According to hypothesis 1, the terms which contain $f_1$ et $f_2$ will disappear, too. So, we have:

$$\begin{aligned}
&\frac{1}{\Delta t}(v^{n+1}-v^n, v^{n+1}-v^n)_{0,\Omega^F} - \frac{1}{\Delta t}(v^n-v^{n-1}, v^{n+1}-v^n)_{0,\Omega^F} \\
&+\frac{1}{\Delta t}(\nu^{n+1}-\nu^n, \nu^{n+1}-\nu^n)_{0,\Omega^S} - \frac{1}{\Delta t}(\nu^n-\nu^{n-1}, \nu^{n+1}-\nu^n)_{0,\Omega^S} \\
&+\nu_{cin} a_F(v^{n+1}-v^n, v^{n+1}-v^n) + \frac{\Delta t}{2\rho} a_S(\nu^{n+1}+\nu^n, \nu^{n+1}-\nu^n) = 0
\end{aligned} \tag{10}$$

The equality $2(a, a-b) = \| a \|^2 - \| b \|^2 + \| a-b \|^2$ is used twice, with $a = v^{n+1} - v^n$ (resp $a = v^n - v^{n-1}$) and $b = \nu^{n+1} - \nu^n$(resp $b = \nu^n - \nu^{n-1}$). From the fact that $a_S$ is symmetrical, it follows that:

$$\begin{aligned}
&\| v^{n+1}-v^n \|^2_{0,\Omega^F} + \| v^{n+1}-2v^n+v^{n-1} \|^2_{0,\Omega^F} \\
&+ \| \nu^{n+1}-\nu^n \|^2_{0,\Omega^S} + \| \nu^{n+1}-2\nu^n+\nu^{n-1} \|^2_{0,\Omega^S} \\
&+2(\Delta t)\nu_{cin} a_F(v^{n+1}-v^n, v^{n+1}-v^n) + \tfrac{(\Delta t)^2}{\rho} a_S(\nu^{n+1}, \nu^{n+1}) \\
&\leq \| v^n - v^{n-1} \|^2_{0,\Omega^F} + \| \nu^n - \nu^{n-1} \|^2_{0,\Omega^S} + \tfrac{(\Delta t)^2}{\rho} a_S(\nu^n, \nu^n)
\end{aligned} \tag{11}$$

Since $0 \leq a_F(v, v)$, we obtain:

$$\begin{aligned}
&\| v^{n+1}-v^n \|^2_{0,\Omega^F} + \| \nu^{n+1}-\nu^n \|^2_{0,\Omega^S} + \frac{(\Delta t)^2}{\rho} a_S(\nu^{n+1}, \nu^{n+1}) \\
&\leq \| v^n - v^{n-1} \|^2_{0,\Omega^F} + \| \nu^n - \nu^{n-1} \|^2_{0,\Omega^S} + \frac{(\Delta t)^2}{\rho} a_S(\nu^n, \nu^n)
\end{aligned} \tag{12}$$

and this implies

$$\begin{aligned}
&\| v^{n+1}-v^n \|^2_{0,\Omega^F} + \| \nu^{n+1}-\nu^n \|^2_{0,\Omega^S} + \frac{(\Delta t)^2}{\rho} a_S(\nu^{n+1}, \nu^{n+1}) \\
&\leq \| v^1 - v^0 \|^2_{0,\Omega^F} + \| \nu^1 - \nu^0 \|^2_{0,\Omega^S} + \frac{(\Delta t)^2}{\rho} a_S(\nu^1, \nu^1)
\end{aligned} \tag{13}$$

According to the hypotheses 2 et 3, we can verify:

$$\left(\| v^{n+1}-v^n \|^2_{0,\Omega^F} + \| \nu^{n+1}-\nu^n \|^2_{0,\Omega^S}\right)^{1/2} \leq (\Delta t) K_3, \qquad \forall n = 0, 1, \ldots \tag{14}$$

Using the triangle inequality, we have:

$$\begin{aligned}
&\left(\| v^{n+1} \|^2_{0,\Omega^F} + \| \nu^{n+1} \|^2_{0,\Omega^S}\right)^{1/2} - \left(\| v^n \|^2_{0,\Omega^F} + \| \nu^n \|^2_{0,\Omega^S}\right)^{1/2} \\
&\leq \left(\| v^{n+1}-v^n \|^2_{0,\Omega^F} + \| \nu^{n+1}-\nu^n \|^2_{0,\Omega^S}\right)^{1/2}
\end{aligned} \tag{15}$$

From the inequalities (14) and (15), it follows:

$$\begin{aligned}
&\left(\| v^{n+1} \|^2_{0,\Omega^F} + \| \nu^{n+1} \|^2_{0,\Omega^S}\right)^{1/2} \\
&-\left(\| v^n \|^2_{0,\Omega^F} + \| \nu^n \|^2_{0,\Omega^S}\right)^{1/2} \leq (\Delta t) K_3
\end{aligned} \tag{16}$$

We write the inequality (16) for $n = 0, 1, \dots$ and we sum them. It is obtained:

$$\begin{aligned}&\Big(\| v^n \|^2_{0,\Omega^F} + \| \nu^n \|^2_{0,\Omega^S}\Big)^{1/2}\\ -&\Big(\| v^0 \|^2_{0,\Omega^F} + \| \nu^0 \|^2_{0,\Omega^S}\Big)^{1/2} \leq (n\Delta t)K_3\end{aligned} \tag{17}$$

But $n\Delta t \leq T$, so that the conclusion of the first step is established.

**2**$^{\text{nd}}$ **Step:** According to the hypotheses 2 et 3 and since the inequality (13), it follows that:

$$\frac{(\Delta t)^2}{\rho} a_S(\nu^{n+1}, \nu^{n+1}) \leq (\Delta t)^2 K_3 \tag{18}$$

It is known that $a_S$ is $W^2-$elliptic, consequently there exists a constant $K_6$, such that:

$$\| \nu^{n+1} \|_{1,\Omega^S} \leq K_6 \tag{19}$$

If we write the relation (1) for $w^1 = v^{n+1}$ and $w^2 = \nu^{n+1}$, we obtain:

$$\begin{aligned}&\frac{1}{\Delta t}(v^{n+1} - v^n, v^{n+1})_{0,\Omega^F} + \frac{1}{\Delta t}(\nu^{n+1} - \nu^n, \nu^{n+1})_{0,\Omega^S} + \nu_{cin} a_F(v^{n+1}, v^{n+1})\\ &+\frac{1}{\rho} a_S\Big(u_0 + \frac{\Delta t}{2}(\nu^0 + 2(\sum_{i=1}^{n} \nu^i) + \nu^{n+1}), \nu^{n+1}\Big)\\ &= (f_1, v^{n+1})_{0,\Omega^F} + (f_2, \nu^{n+1})_{0,\Omega^S}\end{aligned} \tag{20}$$

The following inequalities hold:

$$\begin{aligned}&\frac{1}{\Delta t}(v^{n+1} - v^n, v^{n+1})_{0,\Omega^F} + \frac{1}{\Delta t}(\nu^{n+1} - \nu^n, \nu^{n+1})_{0,\Omega^S}\\ &\leq \frac{1}{\Delta t} \| v^{n+1} - v^n \|_{0,\Omega^F} \| v^{n+1} \|_{0,\Omega^F} + \frac{1}{\Delta t} \| \nu^{n+1} - \nu^n \|_{0,\Omega^S} \| \nu^{n+1} \|_{0,\Omega^S}\\ &\leq \frac{1}{\Delta t}\Big(\| v^{n+1} - v^n \|^2_{0,\Omega^F} + \| \nu^{n+1} - \nu^n \|^2_{0,\Omega^S}\Big)^{1/2}\\ &\times\Big(\| v^{n+1} \|^2_{0,\Omega^F} + \| \nu^{n+1} \|^2_{0,\Omega^S}\Big)^{1/2} \leq K_7\end{aligned} \tag{21}$$

The Cauchy inequality was used to find out the first inequality of (21), the Cauchy-Buniakowski-Schwartz inequality for the second, the inequality (14) and the conclusion of the first step for the third. Also, according to the first step, we have:

$$\begin{aligned}&(f_1, v^{n+1})_{0,\Omega^F} + (f_2, \nu^{n+1})_{0,\Omega^S}\\ &\leq \| f_1 \|_{0,\Omega^F} \| v^{n+1} \|_{0,\Omega^F} + \| f_2 \|_{0,\Omega^S} \| \nu^{n+1} \|_{0,\Omega^S} \leq K_8\end{aligned} \tag{22}$$

Since $a_S$ is continuous, it follows:

$$\begin{aligned}&a_S\Big(u_0 + \frac{\Delta t}{2}(\nu^0 + 2(\sum_{i=1}^{n} \nu^i) + \nu^{n+1}), \nu^{n+1}\Big)\\ &\leq M \| \nu^{n+1} \|_{1,\Omega^S} \Big(\| u_0 \|_{1,\Omega^S} + (n+1)(\Delta t)K_6\Big)\\ &\leq MK_6\Big(\| u_0 \|_{1,\Omega^S} + TK_6\Big)\end{aligned} \tag{23}$$

From the equality (20) and the inequalities (21)-(23), it results that there exists a constant $K_{10}$, such that:

$$\nu_{cin} a_F(v^{n+1}, v^{n+1}) \leq K_{10}$$

Finally, since $a_F$ is coercive and from the fact that $\| \nu^{n+1} \|_{0,\Omega^S}$ is bounded, the inequality proposed to be solved in the second step holds.

**3rd Step:** Using in the equality (1), the inequalities (14 ), the hypothesis 1, the continuity of $a_F$ and $a_S$, the fact that $\| v^{n+1} \|_{1,\Omega^F}$, $\| \nu^{n+1} \|_{1,\Omega^S}$ and $\| u_0 + \frac{\Delta t}{2}(\nu^0 + 2(\sum_{i=1}^n \nu^i) + \nu^{n+1}) \|_{1,\Omega^S}$ are bounded, we obtain:

$$\begin{aligned} &(\mathbf{div}\ w^1, p^{n+1})_{0,\Omega^F} + \Big(\gamma_0(w^1) - \gamma_\Gamma(w^2), \lambda^{n+1}\Big)_{1/2,\Gamma} \\ &\leq K_{11}(\| w^1 \|^2_{1,\Omega^F} + \| w^2 \|^2_{1,\Omega^S})^{1/2} \qquad \forall w^1 \in W^1, \forall w^2 \in W^2 \end{aligned} \tag{24}$$

Finally, in view of the proposition 2 and from the previous inequality, we have:

$$\| p^{n+1} \|_{0,\Omega^F} + \| \lambda^{n+1} \|_{1/2,\Gamma} \leq K_{12}$$

and the proof is finished. □

## Conclusions

The utilisation of Lagrange multipliers to treat the boundary values on the contact surface is very attractive: the problem can be decouplated and the implementation of the algorithm is easy if we have a software, which can solve the fluid problem and the structure problem, separately.

Also, this method can be used in the case when the contact interface is a free surface.

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