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Existence of an Optimal Control for a Nonlinear Fluid-Cable Interaction Problem *

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Abstract. A three-dimensional fluid-cable interaction is studied. The fluid is governed by the Stokes equations and the cable is governed by the beam equations without shearing stress. Only steady equations are studied in this paper. The fluid equations are described using arbitrary lagrangian eulerian coordinates.

The contact surface between fluid and cable is unknown a priori, therefore it is a free boundary like problem.

The fluid-cable interaction is modeled by an optimal control system with Neumann like boundary control and Dirichlet like boundary observation. The control appears also in the coefficients of the fluid equations.

It's a nonlinear and non-convex optimal control problem.

The existence of a solution is proved.

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Key words. fluid-structure interaction, free boundary problem, optimal control, Neumann boundary control, Dirichlet boundary observation, control in the coefficients, arbitrary lagrangian eulerian coordinates

1 Introduction

We study the behavior of a three-dimensional cable under the action of an external flow.

The real system to be modeled is the behavior of an electric cable with fixed extremities under the wind action. We are interested by the displacement of the cable and by the velocity and the pressure of the fluid.

The contact surface between fluid and cable is unknown a priori, therefore it is a free boundary like problem.

We suppose that the fluid is governed by the Stokes equations and the cable is governed by the beam equations without shearing stress. Only steady equations will be studied in this paper.

The fluid and cable equations are coupled via two boundary conditions: equality of the fluid's and cable's velocities at the contact surface (which is a Dirichlet like boundary condition) and equality of the forces at the contact surface (which is a Neumann like boundary condition).

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The coupled fluid-cable problem is modeled by an optimal control variational system. It's a Neumann like boundary control with Dirichlet like boundary observation. The control appears also in the coefficients of the fluid equations.

This mathematical model permits to solve numerically the coupled fluid-cable problem via partitioned procedures (i.e. in a decoupled way, more precisely the fluid and the cable equations are solved separately).

The aim of this paper is to prove the existence of an optimal control for this fluid-cable interaction problem.

2 Notations

Let us consider a cable of cross section S. We assume that $S \subseteq \mathbb{R}^2$ has the following properties: non-empty, open, bounded, connected, with Lipschitz boundary and $(0,0) \in S$.

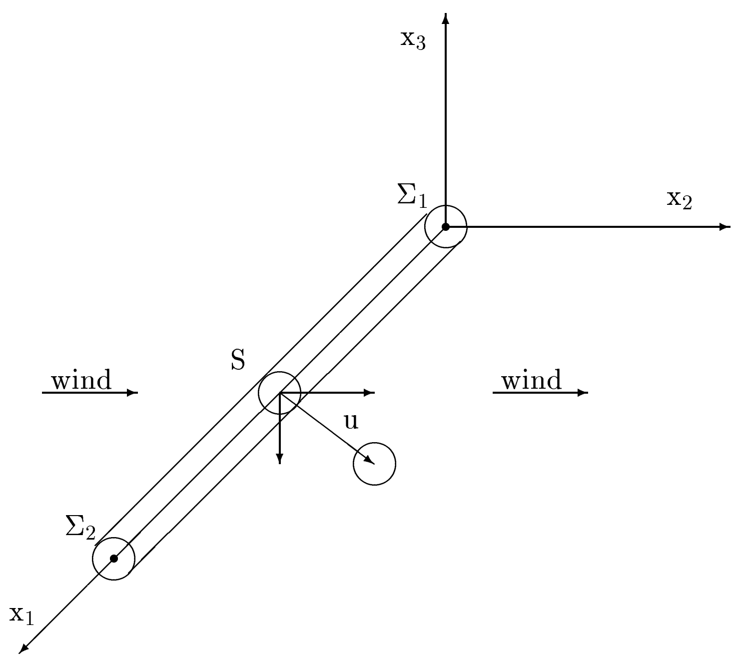

Figure 1: The geometrical configuration of the fluid-cable interaction

The displacement of the cable will be described using the displacement of the median thread noted here by:

$$u = (u_1, u_2, u_3) : [0, L] \to \mathbb{R}^3.$$

For instant, we assume that $u_1 = 0$.

The three-dimensional domain occupied by the cable is

$$\Omega_u^S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 ; x_1 \in]0, L[\, , (x_2 - u_2(x_1), x_3 - u_3(x_1)) \in S \right\} \tag{2.1}$$

and the domain occupied by the fluid is

$$\Omega_u^F = \mathbb{R}^3 \setminus \overline{\Omega_u^S}. \tag{2.2}$$

The contact surface between fluid and cable is

$$\Gamma_u = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 ; x_1 \in]0, L[\, , (x_2 - u_2(x_1), x_3 - u_3(x_1)) \in \partial S \right\} \tag{2.3}$$

which is the free boundary of our problem.

The extremities of the cable are noted:

$$\begin{aligned} \Sigma_1 &= \left\{ (0, x_2, x_3) \in \mathbb{R}^3 ; (x_2, x_3) \in S \right\}, \\ \Sigma_2 &= \left\{ (L, x_2, x_3) \in \mathbb{R}^3 ; (x_2, x_3) \in S \right\} \end{aligned} \tag{2.4}$$

which are fixed.

3 Variational formulation for the cable equations

Now, we present the variational formulation for the cable equations. We have supposed that the cable is governed by the beam equations without shearing stress (see [4]).

Let $D_2 \in \mathbb{R}_+^*$ be given. We set

$$\left\{ \begin{array}{c} a_S : H_0^2(]0, L[) \times H_0^2(]0, L[) \to \mathbb{R} \\ a_S(\phi, \psi) = D_2 \cdot \int_{]0,L[} \frac{d^2\phi}{dx_1^2}(x_1) \frac{d^2\psi}{dx_1^2}(x_1) \, dx_1 \end{array} \right. \tag{3.1}$$

The form a_S is evidently symmetric, bilinear, continuous. In addition, applying the Poincaré inequality (see [8, vol. 3, chap. IV, p. 920]), we obtain that a_S is $H_0^2(]0, L[)$-elliptic.

Let $H^{-2}(]0, L[)$ be the dual of the $H_0^2(]0, L[)$. In this section, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-2}(]0, L[)$ and $H_0^2(]0, L[)$.

As a simple consequence of the Lax-Milgram Theorem (see [8, vol. 4, chap. VII, p. 1217]), we have the following result:

Proposition 1 *Let $f_i^S \in H^{-2}(]0, L[)$ and $\eta_i \in L^2(]0, L[)$ for $i = 2, 3$. Then, the problem: Find u_2, u_3 in $H_0^2(]0, L[)$ such that*

$$a_S(u_i, \psi) = \int_{]0,L[} \eta_i(x_1) \psi(x_1) \, dx_1 + \left\langle f_i^S, \psi \right\rangle, \quad \forall \psi \in H_0^2(]0, L[), \ i = 2, 3 \tag{3.2}$$

has a unique solution.

In order to couple the 3D Stokes equations of the fluid with the beam equations described using the median thread, which is a curve in $\mathbb{R}^3$, we shall need the following result:

Proposition 2 *There is a linear and continuous operator D mapping $L^2(]0, L[\times \partial S)$ onto $L^2(]0, L[)$ such that:*

$$(Dg)(x_1) = \int_{\partial S} g(x_1, \sigma) \, d\sigma, \qquad a.e. \ x_1 \in]0, L[\, . \tag{3.3}$$

Proof. Let g be an element of $L^2\left(]0,L[\times\partial S\right)$. Applying the Fubini's Theorem (see [12, p. 140] for example), we have $g\left(x_1,\cdot\right)\in L^1\left(\partial S\right)$, a.e. $x_1\in\,]0,L[$ and the map

$$x_1\in\,]0,L[\,\mapsto\int_{\partial S}g\left(x_1,\sigma\right)d\sigma$$

is Lebesgue measurable.

From the Schwarz inequality (see [12, p. 62]), it follows that

$$\left(\int_{\partial S}g\left(x_1,\sigma\right)d\sigma\right)^2\leq\left(\int_{\partial S}1d\sigma\right)\int_{\partial S}g^2\left(x_1,\sigma\right)d\sigma,\qquad a.e.\ x_1\in\,]0,L[\,.$$

Integrating the above inequality on $]0,L[$, we have

$$\int_{]0,L[}\left(\int_{\partial S}g\left(x_1,\sigma\right)d\sigma\right)^2\leq L\left(\int_{\partial S}1d\sigma\right)\int_{]0,L[}\left(\int_{\partial S}g^2\left(x_1,\sigma\right)d\sigma\right)dx_1.$$

Using once again the Fubini's Theorem, we obtain

$$\int_{]0,L[}\left(\int_{\partial S}g^2\left(x_1,\sigma\right)d\sigma\right)dx_1=\int_{]0,L[\times\partial S}g^2\left(x_1,\sigma\right)d\sigma dx_1=\|g\|^2_{0,]0,L[\times\partial S}$$

therefore $Dg\in L^2\left(]0,L[\right)$ and

$$\|Dg\|^2_{0,]0,L[}\leq L\left(\int_{\partial S}1d\sigma\right)\|g\|^2_{0,]0,L[\times\partial S}\,. \tag{3.4}$$

The operator D is linear from the linearity of the Lebesgue integral.

The inequality (3.4) implies the continuity of the linear operator D. $\square$

4 Mixed formulation for the fluid equations in moving exterior domain

Let u_2 and u_3 be the solutions of the equation (3.2) and Ω_u^F be the domain occupied by the fluid given by the relations (2.2) and (2.1).

Let us consider the Sobolev space with weights:

$$W^1\left(\Omega_u^F\right)=\left\{w\in\mathcal{D}'\left(\Omega_u^F\right);\ \frac{w\left(x\right)}{\left(1+\|x\|^2\right)^{1/2}}\in L^2\left(\Omega_u^F\right);\ \frac{\partial w}{\partial x_i}\in L^2\left(\Omega_u^F\right);\ i=1,2,3\right\}$$

where $\|x\|=\left(\sum_{i=1}^3x_i^2\right)^{1/2}$ is the eulerian norm in $\mathbb{R}^3$.

We set

$$|w|_{1,\Omega_u^F}=\left(\sum_{i=1}^3\int_{\Omega_u^F}\left|\frac{\partial w}{\partial x_i}\left(x\right)\right|^2dx\right)^{1/2}$$

which is a semi-norm and

$$\|w\|_{1,\Omega_u^F}=\left(\int_{\Omega_u^F}\frac{\left|w\left(x\right)\right|^2}{1+\|x\|^2}dx+|w|^2_{1,\Omega_u^F}\right)^{1/2}$$

which is a norm.

We denote by $W_0^1\left(\Omega_u^F\right)$ the closure of $\mathcal{D}\left(\Omega_u^F\right)$ in $W^1\left(\Omega_u^F\right)$ for the norm $\|\cdot\|_{1,\Omega_u^F}$.

From the Theorem 1 [8, vol. 6, chap. XI B, p. 650], we have that the semi-norm $|\cdot|_{1,\Omega_u^F}$ is a norm for the spaces $W^1\left(\Omega_u^F\right)$ and $W_0^1\left(\Omega_u^F\right)$. Moreover, it's equivalent to $\|\cdot\|_{1,\Omega_u^F}$.

In view of the Remark 2 [8, vol. 6, chap. XI B, p. 651], the spaces $W^1\left(\Omega_u^F\right)$ and $W_0^1\left(\Omega_u^F\right)$ are identical with the Beppo-Levi spaces $BL\left(\Omega_u^F\right)$ and $\hat{\mathcal{D}}^1\left(\Omega_u^F\right)$ respectively. So, the spaces $W^1\left(\Omega_u^F\right)$ and $W_0^1\left(\Omega_u^F\right)$ are the Hilbert spaces for the scalar product:

$$(\phi,\ \psi)=\sum_{i=1}^{3}\int_{\Omega_u^F}\frac{\partial\phi}{\partial x_i}\frac{\partial\psi}{\partial x_i}dx$$

(see also the Remark 7 [8, vol. 5, chap. IX A, p. 264]).

In view of the Sobolev Embedding Theorem (see [1]), we have

$$H_0^2\left(]0,L[\right)\hookrightarrow\mathcal{C}^1\left([0,L]\right)$$

therefore the boundary Γ_u is Lipschitz, so we can define the space $H^{1/2}\left(\Gamma_u\right)$.

From the Theorem 2 [8, vol. 6, chap. XI B, p. 652], there exists the trace application mapping $W^1\left(\Omega_u^F\right)$ onto $H^{1/2}\left(\Gamma_u\right)$ denoted by

$$w\longrightarrow w_{|\Gamma_u}$$

which is continuous and surjective and

$$W_0^1\left(\Omega_u^F\right)=\left\{w\in W^1\left(\Omega_u^F\right);\ w_{|\Gamma_u\cup\overline{\Sigma}_1\cup\overline{\Sigma}_2}=0\right\}$$

Let us consider the following Hilbert space:

$$W_u=\left\{w\in\left(W^1\left(\Omega_u^F\right)\right)^3;\ w=0\text{ on }\overline{\Sigma}_1\cup\overline{\Sigma}_2\right\}$$

equipped with the scalar product

$$(v,w)=\sum_{i,j=1}^{3}\int_{\Omega_u^F}\frac{\partial v_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}dx$$

where $v=(v_1,v_2,v_3)$ and $w=(w_1,w_2,w_3)$ are in W_u.

Also, let us consider the Hilbert space:

$$Q_u=L^2\left(\Omega_u^F\right)$$

equipped with the habitual scalar product.

We use the notation $div\ w=\frac{\partial w_1}{\partial x_1}+\frac{\partial w_2}{\partial x_2}+\frac{\partial w_3}{\partial x_3}$ for all $w=(w_1,w_2,w_3)$ in $\left(W^1\left(\Omega_u^F\right)\right)^3$.

Lemma 1 *For all u_2 and u_3 in $H_0^2\left(]0,L[\right)$, the operator div mapping W_u onto Q_u is surjectif.*

Proof. Let u_2 and u_3 be in $H_0^2\left(]0,L[\right)$.

Let us denote:

$$L_0^2\left(\Omega_u^F\right)=\left\{q\in L^2\left(\Omega_u^F\right);\ \int_{\Omega_u^F}q\ dx=0\right\}$$

It is known (see [8, vol. 5, chap IX A, p. 267, Remark 8] for example) that $div\left(W_0^1\left(\Omega_u^F\right)\right)^3 = L_0^2\left(\Omega_u^F\right)$. Evidently we have $\left(W_0^1\left(\Omega_u^F\right)\right)^3 \subseteq W_u$, so

$$L_0^2\left(\Omega_u^F\right) \subseteq div\left(W_u\right) \subseteq L^2\left(\Omega_u^F\right) \tag{4.1}$$

Since the operator div is linear, we obtain that $div\left(W_u\right)$ is a vectorial subspace.

Knowing that the co-dimension of $L_0^2\left(\Omega_u^F\right)$ in $L^2\left(\Omega_u^F\right)$ is one, the inclusions (4.1) imply: $div\left(W_u\right) = L_0^2\left(\Omega_u^F\right)$ or $div\left(W_u\right) = L^2\left(\Omega_u^F\right)$.

In order to finish the proof of this Lemma, we shall prove that $div\left(W_u\right) \neq L_0^2\left(\Omega_u^F\right)$.

To obtain a contradiction, we suppose that

$$div\left(W_u\right) = L_0^2\left(\Omega_u^F\right) = div\left(W_0^1\left(\Omega_u^F\right)\right)^3 \tag{4.2}$$

Let w be in W_u such that $\sum_{i=1}^3 \int_{\Gamma_u} w_i n_i \, d\sigma > 0$, where n is the unit outward normal to Γ_u.

From (4.2) we obtain that there is ψ in $\left(W_0^1\left(\Omega_u^F\right)\right)^3$ such that $div\ \psi = div\ w$.

From the Green's formula we have

$$0 = \int_{\Omega_u^F} div\left(w - \psi\right)\ dx = \sum_{i=1}^3 \int_{\overline{\Sigma}_1 \cup \overline{\Sigma}_2 \cup \Gamma_u} \left(w_i - \psi_i\right) n_i \, d\sigma = \sum_{i=1}^3 \int_{\Gamma_u} w_i n_i \, d\sigma > 0$$

and we have obtained a contradiction. Then the proof of this Lemma is finished. □

Remark. As good as for a bounded domain, the Green's formula holds for an exterior domain, i.e. complement of a compact (see [8, vol. 6, chap. XI B, p. 694]).

We set

$$\begin{cases} a_F : \left(H_0^2\left(]0, L[\right)\right)^3 \times W_u \times W_u \to \mathbb{R} \\ a_F\left(u, v, w\right) = \displaystyle\sum_{i,j=1}^3 \int_{\Omega_u^F} \frac{\partial v_i}{\partial x_j} \frac{\partial w_i}{\partial x_j} dx \end{cases} \tag{4.3}$$

and

$$\begin{cases} b_F : \left(H_0^2\left(]0, L[\right)\right)^3 \times W_u \times Q_u \to \mathbb{R} \\ b_F\left(u, w, q\right) = -\displaystyle\int_{\Omega_u^F} \left(div\ w\right) q \, dx \end{cases} \tag{4.4}$$

Proposition 3 *For all u_2, u_3 in $H_0^2\left(]0, L[\right)$, $\lambda = \left(\lambda_1, \lambda_2, \lambda_3\right)$ in $\left(L^2\left(\Gamma_u\right)\right)^3$, the problem: Find $(v,\ p) \in W_u \times Q_u$ such that*

$$\begin{cases} a_F\left(u, v, w\right) + b_F\left(u, w, p\right) = \displaystyle\sum_{i=1}^3 \int_{\Gamma_u} \lambda_i w_i \, d\sigma, & \forall w \in W_u \\ b_F\left(u, v, q\right) = 0, & \forall q \in Q_u \end{cases} \tag{4.5}$$

has a unique solution.

Proof. For all u_2, u_3 in $H_0^2\left(]0, L[\right)$, the bilinear form $a_F\left(u, \cdot, \cdot\right)$ is continuous and W_u-elliptic for the norm $\left|\cdot\right|_{1,\Omega_u^F}$.

Using the Lemma 1 and a property of the surjectif operators [6, Theorem II.19, p. 29], we obtain that the inf-sup condition holds for the bilinear form $b_F\left(u, \cdot, \cdot\right)$.

Now, the conclusion of our proposition is a simple consequence of the results of Babuska [3] and Brezzi [7]. □

Remark. The system (4.5) represents the mixed formulation for the Stokes equations in an exterior domain: v and p are the velocity and the pressure of the fluid, λ are the forces on the surface Γ_0.

5 Mixed formulation for the fluid equations in a fixed exterior domain

In order to obtain the mixed formulation for the fluid equations in a fixed exterior domain, the arbitrary lagrangian eulerian coordinates have been used. The formulation in a fixed domain permits us to obtain the existence of the solution for the cable-fluid coupled problem.

We set

$$\Omega_0^S = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 ;\ x_1 \in]0, L[\,,\ (x_2, x_3) \in S \right\},$$
$$\Omega_0^F = \mathbb{R}^3 \setminus \overline{\Omega_0^S},$$
$$\Gamma_0 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 ;\ x_1 \in]0, L[\,,\ (x_2, x_3) \in \partial S \right\}.$$

Let $u_2,\ u_3$ in $H_0^2\left(]0, L[\right)$ be given. We have $H_0^2\left(]0, L[\right) \hookrightarrow \mathcal{C}^1\left([0, L]\right)$ and we extend $u_2,\ u_3$ by zero in the exterior of the interval $[0, L]$. Without risk of confusion, we use the same notations $u_2,\ u_3$ for the extended functions, so

$$u_2(x_1) = u_3(x_1) = 0, \quad \forall x_1 \notin [0, L]$$

Therefore we have $u_i \in \mathcal{C}^1(\mathbb{R})$ and we denote by u_i' the first derivate of u_i for $i = 2, 3$.

Let us consider the following one-to-one continuous differentiable transformation:

$$\begin{gathered} T_u : \mathbb{R}^3 \to \mathbb{R}^3 \\ T_u(\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) = (\widehat{x}_1, \widehat{x}_2 + u_2(\widehat{x}_1), \widehat{x}_3 + u_3(\widehat{x}_1)) \end{gathered} \tag{5.1}$$

which admits the continuous differentiable inverse below:

$$\begin{gathered} T_u^{-1} : \mathbb{R}^3 \to \mathbb{R}^3 \\ T_u^{-1}(x_1, x_2, x_3) = (x_1, x_2 - u_2(x_1), x_3 - u_3(x_1)) \end{gathered} \tag{5.2}$$

We have

$$\begin{gathered} T_u\left(\Omega_0^F\right) = \Omega_u^F \\ T_u(\Gamma_0) = \Gamma_u \\ T_u(\widehat{x}) = \widehat{x},\ \forall \widehat{x} \in \overline{\Sigma}_1 \cup \overline{\Sigma}_2 \end{gathered}$$

We denote by

$$\begin{aligned} Jac\,T_u(\widehat{x}) &= \begin{pmatrix} 1 & 0 & 0 \\ u_2'(\widehat{x}_1) & 1 & 0 \\ u_3'(\widehat{x}_1) & 0 & 1 \end{pmatrix} \\ Jac\,T_u^{-1}(x) &= \begin{pmatrix} 1 & 0 & 0 \\ -u_2'(x_1) & 1 & 0 \\ -u_3'(x_1) & 0 & 1 \end{pmatrix} \end{aligned}$$

the Jacobi matrices of the transformations T_u and T_u^{-1} respectively.

We set $T_u(\widehat{x}) = x$, where $x = (x_1, x_2, x_3) \in \Omega_u^F$ and $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \widehat{x}_3) \in \Omega_0^F$ and $T_u(\widehat{\sigma}) = \sigma$ where $\sigma \in \Gamma_u$ and $\widehat{\sigma} \in \Gamma_0$.

If A is a square matrix, we denote by $det(A)\,,\ A^{-1},\ A^t$ its determinant, the inverse and the transpose matrix, respectively.

Lemma 2 *We have:*

$$\left\{ \begin{aligned} \phi \in L^1\left(\Omega_u^F\right) \quad &\Leftrightarrow \quad \widehat{\phi} = \phi \circ T_u \in L^1\left(\Omega_0^F\right) \\ \int_{\Omega_u^F} \phi(x)\, dx \quad &= \quad \int_{\Omega_0^F} \widehat{\phi}(\widehat{x})\, d\widehat{x} \end{aligned} \right. \tag{5.3}$$

$$\left\{ \begin{aligned} \widehat{\phi} \in L^1(\Gamma_0) \quad &\Rightarrow \quad \phi\, \omega_u \in L^1(\Gamma_u) \\ \int_{\Gamma_0} \widehat{\phi}(\widehat{\sigma})\, d\widehat{\sigma} \quad &= \quad \int_{\Gamma_u} \phi(\sigma)\, \omega_u(\sigma)\, d\sigma \end{aligned} \right. \tag{5.4}$$

where $\phi = \left(\widehat{\phi} \circ T_u^{-1}\right)$, $\omega_u(\sigma) = \left\| det\left(Jac\, T_u^{-1}(\sigma)\right)\left(\left(Jac\, T_u^{-1}(\sigma)\right)^{-1}\right)^t n(\sigma)\right\|_{\mathbb{R}^3}$ *and* $n(\sigma)$ *is the unit outward normal to* Γ_u *in* σ*;*

$$\left\{ \begin{array}{ccc} \phi \in W^1\left(\Omega_u^F\right) & \Leftrightarrow & \widehat{\phi} = \phi \circ T_u \in W^1\left(\Omega_0^F\right) \\ \begin{pmatrix} \dfrac{\partial \phi}{\partial x_1}(x) \\ \dfrac{\partial \phi}{\partial x_2}(x) \\ \dfrac{\partial \phi}{\partial x_3}(x) \end{pmatrix} & = & \left(\left(Jac\, T_u(\widehat{x})\right)^{-1}\right)^t \begin{pmatrix} \dfrac{\partial \widehat{\phi}}{\partial \widehat{x}_1}(\widehat{x}) \\ \dfrac{\partial \widehat{\phi}}{\partial \widehat{x}_2}(\widehat{x}) \\ \dfrac{\partial \widehat{\phi}}{\partial \widehat{x}_3}(\widehat{x}) \end{pmatrix} \end{array} \right. \tag{5.5}$$

Proof. Since $det\left(Jac\, T_u(\widehat{x})\right) = det\left(Jac\, T_u^{-1}(x)\right) = 1$ for all x and $\widehat{x}$ in $\mathbb{R}^3$, the assertion (5.3) is a consequence of the change-of-variable formula for the unbounded domains (see [2, Theorem VIII.5.1, p. 352]).

The proof of (5.4) can be founded in [13, Prop. 2.47, p. 78].

Let us prove (5.5). Let $\phi \in W^1\left(\Omega_u^F\right)$ be given.
From the change-of-variable formula (5.3), we have

$$\int_{\Omega_u^F} \frac{|\phi(x)|^2}{1 + \|x\|^2} dx = \int_{\Omega_0^F} \frac{\left|\widehat{\phi}(\widehat{x})\right|^2}{1 + \|\widehat{x}\|^2} d\widehat{x}$$

Let $\psi \in \mathcal{D}\left(\Omega_0^F\right)$ and $supp\ \psi \subseteq \mathcal{O}$ where $\mathcal{O}$ is a open and bounded set in Ω_0^F.
Since T_u is a diffeomorphism, we have than $T_u(\mathcal{O})$ is a open and bounded set in Ω_u^F.
From the Remark 7 [8, vol. 5, chap IX A, p. 264], we have $\phi \in H^1\left(T_u(\mathcal{O})\right)$ and using [6, Prop. IX.6, p. 156], we obtain that $\widehat{\phi} \in H^1(\mathcal{O})$ and

$$\int_{\mathcal{O}} (\phi \circ T_u)(\widehat{x}) \frac{\partial \psi}{\partial \widehat{x}_j}(\widehat{x})\, d\widehat{x} = -\sum_{i=1}^{3} \int_{\mathcal{O}} \frac{\partial \phi}{\partial x_i}\left(T_u(\widehat{x})\right) \frac{\partial T_{u,i}}{\partial \widehat{x}_j}(\widehat{x})\, \psi(\widehat{x})\, d\widehat{x}$$

The above equality holds also if we change $\mathcal{O}$ by Ω_0^F because $supp\ \psi \subseteq \mathcal{O} \subseteq \Omega_0^F$ then the equality from the second row of the system (5.5) holds.

Since $Jac\, T_u$ is in $\left(L^\infty\left(\Omega_0^F\right)\right)^9$, we have that $\frac{\partial \widehat{\phi}}{\partial \widehat{x}_j}$ are in $L^2\left(\Omega_0^F\right)$ for $j = 1, 2, 3$, so $\widehat{\phi} \in W^1\left(\Omega_0^F\right)$.
□

Let us consider the following Hilbert space:

$$\widehat{W} = \left\{\widehat{w} \in \left(W^1\left(\Omega_0^F\right)\right)^3 ;\ \widehat{w} = 0 \text{ on } \overline{\Sigma}_1 \cup \overline{\Sigma}_2\right\}$$

equipped with the scalar product

$$(\widehat{v}, \widehat{w}) = \sum_{i,j=1}^{3} \int_{\Omega_0^F} \frac{\partial \widehat{v}_i}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_j} d\widehat{x} \tag{5.6}$$

where $\widehat{v} = (\widehat{v}_1, \widehat{v}_2, \widehat{v}_3)$ and $\widehat{w} = (\widehat{w}_1, \widehat{w}_2, \widehat{w}_3)$ are in $\widehat{W}$.

Also, let us consider the Hilbert space:

$$\widehat{Q} = L^2\left(\Omega_0^F\right)$$

equipped with the habitual scalar product.

We set

$$\widehat{a}_F : \left(H_0^2\left(]0, L[\right)\right)^3 \times \widehat{W} \times \widehat{W} \to \mathbb{R}$$
$$\widehat{a}_F\left(u, \widehat{v}, \widehat{w}\right)$$
$$= \sum_{i=1}^{3} \int_{\Omega_0^F} \left(\frac{\partial \widehat{v}_i}{\partial \widehat{x}_1} - u_2' \frac{\partial \widehat{v}_i}{\partial \widehat{x}_2} - u_3' \frac{\partial \widehat{v}_i}{\partial \widehat{x}_3}\right)\left(\frac{\partial \widehat{w}_i}{\partial \widehat{x}_1} - u_2' \frac{\partial \widehat{w}_i}{\partial \widehat{x}_2} - u_3' \frac{\partial \widehat{w}_i}{\partial \widehat{x}_3}\right) d\widehat{x} \qquad (5.7)$$
$$+ \sum_{i=1}^{3} \int_{\Omega_0^F} \left(\frac{\partial \widehat{v}_i}{\partial \widehat{x}_2}\frac{\partial \widehat{w}_i}{\partial \widehat{x}_2} + \frac{\partial \widehat{v}_i}{\partial \widehat{x}_3}\frac{\partial \widehat{w}_i}{\partial \widehat{x}_3}\right) d\widehat{x}$$

and

$$\left\{\begin{array}{l} \widehat{b}_F : \left(H_0^2\left(]0, L[\right)\right)^3 \times \widehat{W} \times \widehat{Q} \to \mathbb{R} \\ \widehat{b}_F\left(u, \widehat{w}, \widehat{q}\right) = -\displaystyle\int_{\Omega_0^F} \left(\frac{\partial \widehat{w}_1}{\partial \widehat{x}_1} + \frac{\partial \widehat{w}_2}{\partial \widehat{x}_2} + \frac{\partial \widehat{w}_3}{\partial \widehat{x}_3} - u_2' \frac{\partial \widehat{w}_2}{\partial \widehat{x}_1} - u_3' \frac{\partial \widehat{w}_3}{\partial \widehat{x}_1}\right) \widehat{q}\, d\widehat{x} \end{array}\right. \qquad (5.8)$$

Proposition 4 *For all u_2, u_3 in $H_0^2\left(]0, L[\right)$, $\widehat{\lambda} = \left(\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\right)$ in $\left(L^2\left(\Gamma_0\right)\right)^3$, the problem:*
Find $\left(\widehat{v},\ \widehat{p}\right) \in \widehat{W} \times \widehat{Q}$ such that

$$\left\{\begin{array}{ll} \widehat{a}_F\left(u, \widehat{v}, \widehat{w}\right) + \widehat{b}_F\left(u, \widehat{w}, \widehat{p}\right) = \displaystyle\sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i \widehat{w}_i\, d\widehat{\sigma}, & \forall \widehat{w} \in \widehat{W} \\ \widehat{b}_F\left(u, \widehat{v}, \widehat{q}\right) = 0, & \forall \widehat{q} \in \widehat{Q} \end{array}\right. \qquad (5.9)$$

has a unique solution.

Proof.

Existence: Let u_2, u_3 in $H_0^2\left(]0, L[\right)$ and $\widehat{\lambda} = \left(\widehat{\lambda}_1, \widehat{\lambda}_2, \widehat{\lambda}_3\right)$ in $\left(L^2\left(\Gamma_0\right)\right)^3$ be given. We set $\lambda\left(\sigma\right) = \left(\widehat{\lambda} \circ T_u^{-1}\right)\left(\sigma\right)\omega_u\left(\sigma\right)$ where

$$\omega_u\left(\sigma\right) = \left\| det\left(Jac\, T_u^{-1}\left(\sigma\right)\right)\left(\left(Jac\, T_u^{-1}\left(\sigma\right)\right)^{-1}\right)^t n\left(\sigma\right)\right\|_{\mathbb{R}^3}$$

and $n\left(\sigma\right)$ is the unit outward normal to Γ_u in σ.

From the Lemma 2, we obtain that λ is well defined and $\lambda \in \left(L^2\left(\Gamma_0\right)\right)^3$.

According to the Proposition 3, there exits an unique solution $\left(v, p\right)$ of the mixed system (4.5) and we set $\widehat{v} = v \circ T_u,\ \widehat{p} = p \circ T_u$.

From the Lemma 2, we have that $\widehat{v} \in \widehat{W}$ and $\widehat{p} \in \widehat{Q}$. Using the change-of-variable formula, we obtain that (5.9) holds.

Uniqueness: Let $\left(\widehat{v}^1,\ \widehat{p}^1\right)$ and $\left(\widehat{v}^2,\ \widehat{p}^2\right)$ be two solutions of the (5.9).

We set $v^1 = \widehat{v}^1 \circ T_u^{-1}$, $p^1 = \widehat{p}^1 \circ T_u^{-1}$, $v^2 = \widehat{v}^2 \circ T_u^{-1}$ and $p^2 = \widehat{p}^2 \circ T_u^{-1}$. Using once again the change-of-variable formula, we have that $\left(v^1, p^1\right)$ and $\left(v^2, p^2\right)$ are solutions for (4.5), but this problem has a unique solution, then $\left(v^1, p^1\right) = \left(v^2, p^2\right)$.

It follows that

$$\widehat{v}^1 = \left(\widehat{v}^1 \circ T_u^{-1}\right) \circ T_u = \left(\widehat{v}^2 \circ T_u^{-1}\right) \circ T_u = \widehat{v}^2$$

and in the same way, $\widehat{p}^1 = \widehat{p}^2$.

So, the conclusion of the proposition holds. $\square$

6 Existence of an optimal control for the fluid-cable interaction problem

The coupled fluid-cable problem will be modeled by an optimal control variational system.

In this section, the existence of an optimal control for the fluid-cable interaction problem will be proved.

Let f_i^S in $H^{-2}(]0,L[)$, $i=2,3$ and $\widehat{K}$ compact in $\left(L^2(\Gamma_0)\right)^3$ be given. Let D be the operator defined by the Proposition 2.

We denote by $\widehat{v}_{|\Gamma_0}$ the trace on Γ_0 of $\widehat{v} \in \widehat{W}$ and by $\|\cdot\|_{0,\Gamma_0}$ the habitual norm in $\left(L^2(\Gamma_0)\right)^3$.

We consider the following optimal control problem $\mathcal{P}$:

$$\inf \frac{1}{2} \left\| \widehat{v}_{|\Gamma_0} \right\|_{0,\Gamma_0}^2$$

subject to

$a)$ $\widehat{\lambda} \in \widehat{K}$

$b)$ $u_2, u_3 \in H_0^2(]0,L[)$

$c)$ $a_S(u_i,\psi) = -\int_{]0,L[} \left(D\widehat{\lambda}_i\right)(x_1)\,\psi(x_1)\,dx_1 + \left\langle f_i^S, \psi \right\rangle, \quad \forall \psi \in H_0^2(]0,L[),\ i=2,3$

$d)$ $(\widehat{v},\widehat{p}) \in \widehat{W} \times \widehat{Q}$

$$e) \quad \begin{cases} \widehat{a}_F(u,\widehat{v},\widehat{w}) + \widehat{b}_F(u,\widehat{w},\widehat{p}) = \displaystyle\sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i \widehat{w}_i \, d\widehat{\sigma}, & \forall \widehat{w} \in \widehat{W} \\ \widehat{b}_F(u,\widehat{v},\widehat{q}) = 0, & \forall \widehat{q} \in \widehat{Q} \end{cases}$$

It's an optimal control problem with Neumann like boundary control $(\widehat{\lambda})$ and Dirichlet like boundary observation $(\widehat{v}_{|\Gamma_0})$. The control appears also in the coefficients of the fluid equations (relation e)).

The relation a) represents the control constraint and the second relation of the system e) represents the state constraint.

This mathematical model permits to solve numerically the coupled fluid-cable problem via partitioned procedures (i.e. in a decoupled way, more precisely the fluid and the cable equations are solved separately).

The relations b) and c) represent the cable equations and the relations d) and e) represent the fluid equations.

In the classical approaches, the fluid and structure equations are coupled by two boundary conditions: equality of the fluid's and structure's velocities at the contact surface (which is a Dirichlet like boundary condition) and equality of the forces at the contact surface (which is a Neumann like boundary condition).

In our approach, we start with a guess for the contact forces (step a). The displacement of the structure can be computed (steps b and c). We suppose that domain occupied by the fluid is completely determined by the displacement of the structure. Knowing the actual domain of the fluid and the contact forces, we can compute the velocity and the pressure of the fluid (steps d and e).

In this way, the Neumann like contact boundary condition is trivially accomplished: we use the same value $\widehat{\lambda}$ for the contact forces on Γ_0 in the equations c) (for the cable) and in the equations e) (for the fluid).

The Dirichlet like contact boundary condition $\widehat{v}_{|\Gamma_0} = 0$ is treated by the Least Squares Method

$$\inf \frac{1}{2} \left\| \widehat{v}_{|\Gamma_0} \right\|_{0,\Gamma_0}^2$$

We denote by $|\cdot|_{1,\Omega_0^F}$ the norm induced by the scalar product (5.6). We have that $\widehat{W}$ is a Hilbert space for this scalar product.

If A is a matrix, we denote by A^t the transpose matrix and if y is a column vector of $\mathbb{R}^3$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},$$

we denote by y^t the transpose vector $y^t = (y_1, y_2, y_3)$.

Lemma 3 *Let B be a bounded set in $H_0^2(]0, L[)$.*

Then the following inequalities hold:

$$\exists m_B > 0, \forall u_2, u_3 \in B, \forall \widehat{w} \in \widehat{W}, \quad m_B |\widehat{w}|^2_{1,\Omega_0^F} \le \widehat{a}_F(u, \widehat{w}, \widehat{w})$$
$$\exists M_B > 0, \forall u_2, u_3 \in B, \forall \widehat{w}^1, \widehat{w}^2 \in \widehat{W}, \ \widehat{a}_F\left(u, \widehat{w}^1, \widehat{w}^2\right) \le M_B \left|\widehat{w}^1\right|_{1,\Omega_0^F} \left|\widehat{w}^2\right|_{1,\Omega_0^F}$$

Proof. The equality (5.7) can be rewritten in the form

$$\widehat{a}_F(u, \widehat{v}, \widehat{w}) = \sum_{i=1}^3 \int_{\Omega_0^F} \left(\frac{\partial \widehat{v}_i}{\partial \widehat{x}_1}, \frac{\partial \widehat{v}_i}{\partial \widehat{x}_2}, \frac{\partial \widehat{v}_i}{\partial \widehat{x}_3}\right) L_u^t(\widehat{x}_1) L_u(\widehat{x}_1) \left(\frac{\partial \widehat{w}_i}{\partial \widehat{x}_1}, \frac{\partial \widehat{w}_i}{\partial \widehat{x}_2}, \frac{\partial \widehat{w}_i}{\partial \widehat{x}_3}\right)^t d\widehat{x} \tag{6.1}$$

where for all $\widehat{x}_1$ in $[0, L]$ we denote:

$$L_u(\widehat{x}_1) = \begin{pmatrix} 1 & -u_2'(\widehat{x}_1) & -u_3'(\widehat{x}_1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \left(\left(Jac\, T_u^{-1}(\widehat{x})\right)^{-1}\right)^t$$

Evidently, L is an invertible matrix and we have:

$$L_u^{-1}(\widehat{x}_1) = \begin{pmatrix} 1 & u_2'(\widehat{x}_1) & u_3'(\widehat{x}_1) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We denote by $\|y\| = (y^t y)^{1/2}$ the euclidean norm of $\mathbb{R}^3$. Now, we evaluate the euclidean norm of the matrix L^{-1}.

$$\begin{aligned} \left\|L_u^{-1}(\widehat{x}_1)\right\| &= \max_{\|y\| \le 1} \left\|L_u^{-1}(\widehat{x}_1)\, y\right\| \\ &= \max_{\|y\| \le 1} \sqrt{\left(y_1 + u_2'(\widehat{x}_1)\, y_2 + u_3'(\widehat{x}_1)\, y_3\right)^2 + y_2^2 + y_3^2} \\ &\le \max_{\|y\| \le 1} \sqrt{\left(1 + \left(u_2'(\widehat{x}_1)\right)^2 + \left(u_3'(\widehat{x}_1)\right)^2\right)\left(y_1^2 + y_2^2 + y_3^2\right) + y_2^2 + y_3^2} \\ &\le \sqrt{2 + \left(u_2'(\widehat{x}_1)\right)^2 + \left(u_3'(\widehat{x}_1)\right)^2} \end{aligned}$$

We have

$$\begin{aligned} |u_i'(\widehat{x}_1)| &= \left|\int_0^{\widehat{x}_1} u_i''(s)\, ds\right| \le \int_0^{\widehat{x}_1} |u_i''(s)|\, ds \\ &\le \int_0^L |u_i''(s)|\, ds \le \sqrt{L}\left(\int_0^L \left(u_i''(s)\right)^2 ds\right)^{1/2} \le \sqrt{L}\, \|u_i\|_{2,]0,L[} \end{aligned} \tag{6.2}$$

The set B is bounded in $H_0^2\left(]0,L[\right)$, then

$$\exists \alpha_B > 0, \forall u_2, u_3 \in B, \forall \widehat{x}_1 \in [0,L], \quad \left\| L_u^{-1}(\widehat{x}_1) \right\| \leq \sqrt{\alpha_B}$$

It follows that

$$\begin{gathered}\forall u_2, u_3 \in B, \forall \widehat{x}_1 \in [0,L], \forall y \in \mathbb{R}^3, \\ \left\| L_u(\widehat{x}_1)\, y \right\|^2 = y^t L_u^t(\widehat{x}_1)\, L_u(\widehat{x}_1)\, y \geq \frac{1}{\alpha_B} \left\| y \right\|^2\end{gathered}$$

Using the last inequality, we obtain from (6.1) the following relation:

$$\begin{gathered}\forall u_2, u_3 \in B, \forall \widehat{w} \in \widehat{W}, \\ \frac{1}{\alpha_B} \left| \widehat{w} \right|_{1,\Omega_0^F} = \frac{1}{\alpha_B} \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\Omega_0^F} \left| \frac{\partial \widehat{w}_i}{\partial \widehat{x}_j}(\widehat{x}) \right|^2 d\widehat{x} \leq \widehat{a}_F(u, \widehat{w}, \widehat{w})\end{gathered}$$

In the same way, we have

$$\forall u_2, u_3 \in B, \forall \widehat{x}_1 \in [0,L], \quad \left\| L_u(\widehat{x}_1) \right\| \leq \sqrt{\alpha_B}$$

and then

$$\begin{gathered}\forall u_2, u_3 \in B, \forall \widehat{x}_1 \in [0,L], \forall y, z \in \mathbb{R}^3, \\ z^t L_u^t(\widehat{x}_1)\, L_u(\widehat{x}_1)\, y \leq \alpha_B z^t y\end{gathered}$$

Using the last inequality, we obtain from (6.1) the following relation:

$$\begin{gathered}\forall u_2, u_3 \in B, \forall \widehat{w}^1, \widehat{w}^2 \in \widehat{W}, \\ \widehat{a}_F\left(u, \widehat{w}^1, \widehat{w}^2\right) \leq \alpha_B \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{\Omega_0^F} \frac{\partial \widehat{w}_i^1}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i^2}{\partial \widehat{x}_j} d\widehat{x} \leq \alpha_B \left| \widehat{w}^1 \right|_{1,\Omega_0^F} \left| \widehat{w}^2 \right|_{1,\Omega_0^F}\end{gathered}$$

We set $m_B = 1/\alpha_B$, $M_B = \alpha_B$ and then the proof of this lemma is finished. □

Lemma 4 *Let u_2 and u_3 be given in $H_0^2\left(]0,L[\right)$. Then*

$$\exists \delta_u > 0, \forall \widehat{w} \in \widehat{W}, \forall \widehat{q} \in \widehat{Q}, \quad \delta_u \left| \widehat{w} \right|_{1,\Omega_0^F} \left\| \widehat{q} \right\|_{0,\Omega_0^F} \leq \widehat{b}_F(u, \widehat{w}, \widehat{q})$$

Proof. Using the change-of-variable formula (5.5), we obtain that

$$\begin{aligned}\frac{\partial w_1}{\partial x_1}(x) &= \frac{\partial \widehat{w}_1}{\partial \widehat{x}_1}(\widehat{x}) - u_2'(\widehat{x}_1) \frac{\partial \widehat{w}_1}{\partial \widehat{x}_2}(\widehat{x}) - u_3'(\widehat{x}_1) \frac{\partial \widehat{w}_1}{\partial \widehat{x}_3}(\widehat{x}) \\ \frac{\partial w_2}{\partial x_2}(x) &= \frac{\partial \widehat{w}_2}{\partial \widehat{x}_2}(\widehat{x}) \\ \frac{\partial w_3}{\partial x_3}(x) &= \frac{\partial \widehat{w}_3}{\partial \widehat{x}_3}(\widehat{x})\end{aligned}$$

From the Lemma 1 and the above equalities, we obtain that the operator mapping $\widehat{W}$ onto $\widehat{Q}$

$$\widehat{w} \longmapsto \frac{\partial \widehat{w}_1}{\partial \widehat{x}_1} + \frac{\partial \widehat{w}_2}{\partial \widehat{x}_2} + \frac{\partial \widehat{w}_3}{\partial \widehat{x}_3} - u_2' \frac{\partial \widehat{w}_1}{\partial \widehat{x}_2} - u_3' \frac{\partial \widehat{w}_1}{\partial \widehat{x}_3}$$

is surjectif.

In a standard way, from the property of the surjectif operators [6, Theorem II.19, p. 29], we obtain the conclusion of this Lemma. □

Lemma 5 *Let u_2, u_3, $\overline{u}_2$, $\overline{u}_3$ be given in $H_0^2\left(]0,L[\right)$. We denote $u=(0,u_2,u_3)$ and $\overline{u}=(0,\overline{u}_2,\overline{u}_3)$. Then there exists a constant β not depending upon u, $\overline{u}$ such that*

$$\forall\widehat{w}\in\widehat{W},\forall\widehat{q}\in\widehat{Q},\quad \widehat{b}_F\left(u,\widehat{w},\widehat{q}\right)\geq\widehat{b}_F\left(\overline{u},\widehat{w},\widehat{q}\right)-\beta\left\|u-\overline{u}\right\|_{2,]0,L[}\left|\widehat{w}\right|_{1,\Omega_0^F}\left\|\widehat{q}\right\|_{0,\Omega_0^F}$$

Proof. We have

$$\widehat{b}_F\left(u,\widehat{w},\widehat{q}\right)-\widehat{b}_F\left(\overline{u},\widehat{w},\widehat{q}\right)=-\int_{\Omega_0^F}\left(u_2-\overline{u}_2\right)'\frac{\partial\widehat{w}_1}{\partial\widehat{x}_2}\widehat{q}\,d\widehat{x}-\int_{\Omega_0^F}\left(u_3-\overline{u}_3\right)'\frac{\partial\widehat{w}_1}{\partial\widehat{x}_2}\widehat{q}\,d\widehat{x}$$

Using the inequality (6.2) and after the Cauchy-Schwartz inequality, we obtain

$$\int_{\Omega_0^F}\left(u_i-\overline{u}_i\right)'\frac{\partial\widehat{w}_1}{\partial\widehat{x}_i}\widehat{q}\,d\widehat{x}\leq\sqrt{L}\left\|u_i-\overline{u}_i\right\|_{2,]0,L[}\left\|\frac{\partial\widehat{w}_1}{\partial\widehat{x}_i}\right\|_{0,\Omega_0^F}\left\|\widehat{q}\right\|_{0,\Omega_0^F}\qquad\text{for } i=2,3$$

and the proof of this Lemma is finished. □

Theorem 1 *For all f_i^S in $H_0^2\left(]0,L[\right)$, $i=2,3$ and $\widehat{K}$ compact in $\left(L^2\left(\Gamma_0\right)\right)^3$, the problem $\mathcal{P}$ has at least one optimal solution $\left[\widehat{\lambda}^*,u^*,\widehat{v}^*,\widehat{p}^*\right]$, where $\widehat{\lambda}^*$ is the density of the forces on the contact surface, $u^*=(0,u_2^*,u_3^*)$ is the displacement of the cable, $\widehat{v}^*$ and $\widehat{p}^*$ are the velocity and the pressure of the fluid in the arbitrary lagrangian eulerian coordinates. In order to obtain the velocity and the pressure in the real domain we must use the transformation $v^*=\widehat{v}^*\circ T_{u^*}^{-1}$ and $p^*=\widehat{p}^*\circ T_{u^*}^{-1}$.*

Proof. I) The cost functional of the problem $\mathcal{P}$ is evidently positiv, then there exits a real number d such that

$$\inf\frac{1}{2}\left\|\widehat{v}_{|\Gamma_0}\right\|_{0,\Gamma_0}^2=d\tag{6.3}$$

The observation $\widehat{v}$ was computed from the control $\widehat{\lambda}$ using the relations a) - e) of the problem $\mathcal{P}$.

Let $\left\{\widehat{\lambda}^k\right\}_{k\in\mathbb{N}}$ be a minimizing sequence, i.e.

$$\lim_{k\to\infty}\frac{1}{2}\left\|\widehat{v}_{|\Gamma_0}^k\right\|_{0,\Gamma_0}^2=d\tag{6.4}$$

where $\widehat{v}^k$ was computed from $\widehat{\lambda}^k$ using the following relations:

$a')\quad \widehat{\lambda}^k\in\widehat{K}$

$b')\quad u_2^k,u_3^k\in H_0^2\left(]0,L[\right)$

$c')\quad a_S\left(u_i^k,\psi\right)=-\int_{]0,L[}\left(D\widehat{\lambda}_i^k\right)(x_1)\,\psi\left(x_1\right)dx_1+\left\langle f_i^S,\psi\right\rangle,\quad\forall\psi\in H_0^2\left(]0,L[\right),\ i=2,3$

$d')\quad \left(\widehat{v}^k,\widehat{p}^k\right)\in\widehat{W}\times\widehat{Q}$

$$e')\quad\begin{cases}\widehat{a}_F\left(u^k,\widehat{v}^k,\widehat{w}\right)+\widehat{b}_F\left(u^k,\widehat{w},\widehat{p}^k\right)=\displaystyle\sum_{i=1}^3\int_{\Gamma_0}\widehat{\lambda}_i^k\widehat{w}_i\,d\widehat{\sigma}, & \forall\widehat{w}\in\widehat{W}\\ \widehat{b}_F\left(u^k,\widehat{v}^k,\widehat{q}\right)=0, & \forall\widehat{q}\in\widehat{Q}\end{cases}$$

The set $\widehat{K}$ is compact, then there exits a subsequence of $\left\{\widehat{\lambda}^k\right\}_{k\in\mathbb{N}}$ strongly convergent in $\left(L^2\left(\Gamma_0\right)\right)^3$. Without risk of confusion, we use the same notation $\left\{\widehat{\lambda}^k\right\}_{k\in\mathbb{N}}$ for this subsequence. We denote $\widehat{\lambda}^*$ its limit, so

$$\widehat{\lambda}^k\to\widehat{\lambda}^*\text{ strongly in }\left(L^2\left(\Gamma_0\right)\right)^3$$

II) Let $u^* = (0, u_2^*, u_3^*)$ be the displacement of the cable computed using the variational equations b) - c) for the density of the contact forces $\widehat{\lambda}^*$. Since $\left\{\widehat{\lambda}^k\right\}_{k\in\mathbb{N}}$ is strongly convergent in $\left(L^2(\Gamma_0)\right)^3$ to $\widehat{\lambda}^*$, from b') and c') we obtain that $u^k = \left(0, u_2^k, u_3^k\right)$ is strongly convergent to u^* in $\left(H_0^2(]0,L[)\right)^3$. Consequently, there exists a compact B in $H_0^2(]0,L[)$ such that u_2^k, u_3^k belong to B for all k. Also, u_2^*, u_3^* belong to B.

III) From e') we have

$$\widehat{a}_F\left(u^k, \widehat{v}^k, \widehat{v}^k\right) = \sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i^k \widehat{v}_i^k \, d\widehat{\sigma}$$

and using the Lemma 3 and the Cauchy-Schwartz inequality, we obtain

$$m_B \left|\widehat{v}^k\right|_{1,\Omega_0^F}^2 \leq \left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0} \left\|\widehat{v}_{|\Gamma_0}^k\right\|_{0,\Gamma_0}$$

From the Trace Theorem, we have for all $\widehat{w}$ in $\widehat{W}$

$$\left\|\widehat{w}_{|\Gamma_0}\right\|_{0,\Gamma_0} \leq c\left(\Omega_0^F\right) |\widehat{w}|_{1,\Omega_0^F}$$

where $c\left(\Omega_0^F\right)$ is a constant only depending upon Ω_0^F which is fixed.

It follows that

$$m_B \left|\widehat{v}^k\right|_{1,\Omega_0^F}^2 \leq c\left(\Omega_0^F\right) \left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0} \left|\widehat{v}^k\right|_{1,\Omega_0^F}$$

Since $\widehat{\lambda}^k$ is strongly convergent, then $\left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0}$ is bounded which implies that $\left|\widehat{v}^k\right|_{1,\Omega_0^F}$ is bounded, too.

From the first equality of the system e') and the Lemma 3, we have

$$\begin{aligned}
\widehat{b}_F\left(u^k, \widehat{w}, \widehat{p}^k\right) &= \sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i^k \widehat{w}_i \, d\widehat{\sigma} - \widehat{a}_F\left(u^k, \widehat{v}^k, \widehat{w}\right) \\
&\leq \sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i^k \widehat{w}_i \, d\widehat{\sigma} + M_B \left|\widehat{v}^k\right|_{1,\Omega_0^F} |\widehat{w}|_{1,\Omega_0^F}
\end{aligned}$$

From the Cauchy Schwartz inequality and the trace theorem, it follows

$$\sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i^k \widehat{w}_i \, d\widehat{\sigma} \leq \left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0} \left\|\widehat{w}_{|\Gamma_0}\right\|_{0,\Gamma_0} \leq c\left(\Omega_0^F\right) \left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0} |\widehat{w}|_{1,\Omega_0^F}$$

From the above inequalities, we obtain

$$\widehat{b}_F\left(u^k, \widehat{w}, \widehat{p}^k\right) \leq c\left(\Omega_0^F\right) \left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0} |\widehat{w}|_{1,\Omega_0^F} + M_B \left|\widehat{v}^k\right|_{1,\Omega_0^F} |\widehat{w}|_{1,\Omega_0^F} \tag{6.5}$$

From the Lemmas 4 and 5, we have

$$\begin{aligned}
&\delta_{u^*} |\widehat{w}|_{1,\Omega_0^F} \left\|\widehat{p}^k\right\|_{0,\Omega_0^F} - \beta \left\|u^k - u^*\right\|_{2,]0,L[} |\widehat{w}|_{1,\Omega_0^F} \left\|\widehat{p}^k\right\|_{0,\Omega_0^F} \\
&\leq \widehat{b}_F\left(u^*, \widehat{w}, \widehat{p}^k\right) - \beta \left\|u^k - u^*\right\|_{2,]0,L[} |\widehat{w}|_{1,\Omega_0^F} \left\|\widehat{p}^k\right\|_{0,\Omega_0^F} \\
&\leq \widehat{b}_F\left(u^k, \widehat{w}, \widehat{p}^k\right)
\end{aligned} \tag{6.6}$$

Using the inequalities (6.5) and (6.6), we obtain for all k in $\mathbb{N}$ and $\widehat{w}$ in $\widehat{W}$

$$\begin{aligned}
&\left(\delta_{u^*} - \beta \left\|u^k - u^*\right\|_{2,]0,L[}\right) \left\|\widehat{p}^k\right\|_{0,\Omega_0^F} |\widehat{w}|_{1,\Omega_0^F} \\
&\leq \left(c\left(\Omega_0^F\right) \left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0} + M_B \left|\widehat{v}^k\right|_{1,\Omega_0^F}\right) |\widehat{w}|_{1,\Omega_0^F}
\end{aligned}$$

Since $\delta_{u^*} > 0$ is fixed, $\left|\widehat{v}^k\right|_{1,\Omega_0^F}$ and $\left\|\widehat{\lambda}^k\right\|_{0,\Gamma_0}$ are bounded and

$$\lim_{k\to\mathbb{N}} \left\|u^k - u^*\right\|_{2,]0,L[} = 0$$

we obtain that $\left\|\widehat{p}^k\right\|_{0,\Omega_0^F}$ is bounded.

The spaces $\widehat{W}$ and $\widehat{Q}$ are Hilbert, then there exists a subsequence $\left\{\widehat{v}^{k_l}\right\}_{l\in\mathbb{N}}$ weakly convergent in $\widehat{W}$ and $\left\{\widehat{p}^{k_l}\right\}_{l\in\mathbb{N}}$ weakly convergent in $\widehat{Q}$. We denote by $\widehat{v}^{**}$ and $\widehat{p}^{**}$ the limits of these subsequences.

IV) We have from the previous steps

$$\begin{array}{lcll} \widehat{\lambda}^k & \to & \widehat{\lambda}^* & \text{strongly in } \left(L^2\left(\Gamma_0\right)\right)^3 \\ u^k & \to & u^* & \text{strongly in } \left(H_0^2\left(]0,L[\right)\right)^3 \\ \widehat{v}^{k_l} & \to & \widehat{v}^{**} & \text{weakly in } \widehat{W} \\ \widehat{p}^{k_l} & \to & \widehat{p}^{**} & \text{weakly in } \widehat{Q} \end{array}$$

We denote by $(\widehat{v}^*, \widehat{p}^*)$ the solution of the problem (5.9) computed for the displacement u^* and for the forces $\widehat{\lambda}^*$ on the surface Γ_0.

We shall prove that $\widehat{v}^{**} = \widehat{v}^*$, $\widehat{p}^{**} = \widehat{p}^*$, the whole sequence $\left\{\widehat{v}^k\right\}_{k\in\mathbb{N}}$ is weakly convergent to $\widehat{v}^*$ in $\widehat{W}$ and the whole sequence $\left\{\widehat{p}^k\right\}_{k\in\mathbb{N}}$ is weakly convergent to $\widehat{p}^*$ in $\widehat{Q}$. In order to prove this, we shall show that the following equalities hold:

$$\begin{array}{rl} \forall \widehat{w} \in \widehat{W}, & \lim\limits_{l\to\mathbb{N}} \widehat{a}_F\left(u^{k_l}, \widehat{v}^{k_l}, \widehat{w}\right) = \widehat{a}_F\left(u^*, \widehat{v}^{**}, \widehat{w}\right) \\ \forall \widehat{w} \in \widehat{W}, & \lim\limits_{l\to\mathbb{N}} \widehat{b}_F\left(u^{k_l}, \widehat{w}, \widehat{p}^{k_l}\right) = \widehat{b}_F\left(u^*, \widehat{w}, \widehat{p}^{**}\right) \\ \forall \widehat{q} \in \widehat{Q}, & \lim\limits_{l\to\mathbb{N}} \widehat{b}_F\left(u^{k_l}, \widehat{v}^{k_l}, \widehat{q}\right) = \widehat{b}_F\left(u^*, \widehat{v}^{**}, \widehat{q}\right) \end{array}$$

According to (5.7), we have that $\widehat{a}_F\left(u^{k_l}, \widehat{v}^{k_l}, \widehat{w}\right)$ is a sum of terms like these:

i)
$$\int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{k_l}}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_j} d\widehat{x}, \quad j = 1,2,3$$

ii)
$$\int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{k_l}}{\partial \widehat{x}_1} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_j} \left(u_j^{k_l}\right)' d\widehat{x}, \quad j = 2,3$$

iii)
$$\int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{k_l}}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_1} \left(u_j^{k_l}\right)' d\widehat{x}, \quad j = 2,3$$

iv)
$$\int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{k_l}}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_p} \left(u_j^{k_l}\right)' \left(u_p^{k_l}\right)' d\widehat{x}, \quad j,p = 2,3$$

From the definition of the weak convergence, we have

$$\forall \widehat{w} \in \widehat{W}, \quad \lim_{l\to\mathbb{N}} \int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{k_l}}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_j} d\widehat{x} = \int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{**}}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_j} d\widehat{x}$$

The terms *ii)*, *iii)* and *iv)* have the same form:

$$\int_{\Omega_0^F} \frac{\partial \widehat{v}_i^{k_l}}{\partial \widehat{x}_j} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_p} a^{k_l} d\widehat{x}, \quad j,p = 2,3.$$

Since $u_j^{k_l}$ and $u_p^{k_l}$ are strongly convergent to u_j^* and u_p^* in $H_0^2\left(]0,L[\right)$ respectively, we obtain that $\left(u_j^{k_l}\right)'$ and $\left(u_p^{k_l}\right)'$ are strongly convergent to $\left(u_j^*\right)'$ and $\left(u_p^*\right)'$in $H_0^1\left(]0,L[\right)$ respectively. Easily, it follows that the product $\left(u_j^{k_l}\right)'\left(u_p^{k_l}\right)'$ is strongly convergent to $\left(u_j^*\right)'\left(u_p^*\right)'$ in $H_0^1\left(]0,L[\right)$.

Therefore, we have that the sequence $\left\{a^{k_l}\right\}_{k_l\in\mathbb{N}}$ is strongly convergent in the space $H_0^1\left([0,L]\right)$. We denote by a its limit.

In the following, it will be useful the well known below result:

Lemma 6 *Let X be a reflexive Banach space with dual X'. For all sequence $\left\{w^l\right\}_{l\in\mathbb{N}}$ weakly convergent to w in X and all sequence of linear operators $\left\{A_l\right\}_{l\in\mathbb{N}}$ strongly convergent to A in $\mathcal{L}\left(X,X'\right)$, then the sequence $\left\{A_l w^l\right\}_{l\in\mathbb{N}}$ is weakly convergent to Aw in X'.*

In order to apply this Lemma, let us consider the Hilbert space

$$X=\left\{\phi\in W^1\left(\Omega_0^F\right);\ \phi=0 \text{ on } \overline{\Sigma}_1\cup\overline{\Sigma}_2\right\}$$

equipped with the scalar product

$$(\psi,\phi)_X=\sum_{j=1}^{3}\int_{\Omega_0^F}\frac{\partial\psi}{\partial\widehat{x}_j}\frac{\partial\phi}{\partial\widehat{x}_j}d\widehat{x}$$

and the induced norm $\|\phi\|_X=\sqrt{(\phi,\phi)_X}$.

Also, let us consider the operators $A_l, A\in\mathcal{L}\left(X,X'\right)$ definited by

$$\begin{aligned}\langle A_l\phi,\psi\rangle_{X',X}&=\int_{\Omega_0^F}\frac{\partial\psi}{\partial\widehat{x}_j}\frac{\partial\phi}{\partial\widehat{x}_p}a^{k_l}d\widehat{x},\quad\forall\phi,\psi\in X\\ \langle A\phi,\psi\rangle_{X',X}&=\int_{\Omega_0^F}\frac{\partial\psi}{\partial\widehat{x}_j}\frac{\partial\phi}{\partial\widehat{x}_p}\,a\,d\widehat{x},\quad\forall\phi,\psi\in X.\end{aligned}$$

We have

$$\begin{aligned}&\|(A_l-A)\,\phi\|_X=\sup_{\|\psi\|_X\le 1}\langle(A_l-A)\,\phi,\psi\rangle=\sup_{\|\psi\|_X\le 1}\int_{\Omega_0^F}\frac{\partial\psi}{\partial\widehat{x}_j}\frac{\partial\phi}{\partial\widehat{x}_p}\left(a^{k_l}-a\right)d\widehat{x}\\ &\le\left(\int_{\Omega_0^F}\left(\frac{\partial\phi}{\partial\widehat{x}_p}\right)^2\left(a^{k_l}-a\right)^2d\widehat{x}\right)^{1/2}\le\max_{\widehat{x}_1\in[0,L]}\left|\left(a^{k_l}-a\right)(\widehat{x}_1)\right|\left(\int_{\Omega_0^F}\left(\frac{\partial\phi}{\partial\widehat{x}_p}\right)^2d\widehat{x}\right)^{1/2}\\ &\le\max_{\widehat{x}_1\in[0,L]}\left|\left(a^{k_l}-a\right)(\widehat{x}_1)\right|\|\phi\|_X\,,\qquad\forall\phi\in X\end{aligned}$$

Therefore

$$\|(A_l-A)\|_{\mathcal{L}(X,X')}\le\max_{\widehat{x}_1\in[0,L]}\left|\left(a^{k_l}-a\right)(\widehat{x}_1)\right|$$

But

$$\begin{aligned}&\left|\left(a^{k_l}-a\right)(\widehat{x}_1)\right|=\left|\int_0^{\widehat{x}_1}\left(a^{k_l}-a\right)'(s)\,ds\right|\le\int_0^{\widehat{x}_1}\left|\left(a^{k_l}-a\right)'(s)\right|ds\\ &\le\int_0^L\left|\left(a^{k_l}-a\right)'(s)\right|ds\le\sqrt{L}\left(\int_0^L\left(\left(a^{k_l}-a\right)'(s)\right)^2ds\right)^{1/2}\le\sqrt{L}\left\|a^{k_l}-a\right\|_{1,]0,L[}\end{aligned}$$

then A_l is strongly convergent to A in $\mathcal{L}\left(X,X'\right)$.

Applying the Lemma 6, we obtain that

$$\lim_{l\to\infty}\left\langle A_l \widehat{v}_i^{k_l}, \widehat{w}_i\right\rangle_{X',X} = \left\langle A, \widehat{v}_i^{**}, \widehat{w}_i\right\rangle_{X',X}$$

and consequently

$$\forall \widehat{w} \in \widehat{W}, \quad \lim_{l\to\infty} \widehat{a}_F\left(u^{k_l}, \widehat{v}^{k_l}, \widehat{w}\right) = \widehat{a}_F\left(u^*, \widehat{v}^{**}, \widehat{w}\right)$$

Using the same technique, we obtain

$$\begin{aligned} \forall \widehat{w} \in \widehat{W}, &\quad \lim_{l\to\infty} \widehat{b}_F\left(u^{k_l}, \widehat{w}, \widehat{p}^{k_l}\right) = \widehat{b}_F\left(u^*, \widehat{w}, \widehat{p}^{**}\right) \\ \forall \widehat{q} \in \widehat{Q}, &\quad \lim_{l\to\infty} \widehat{b}_F\left(u^{k_l}, \widehat{v}^{k_l}, \widehat{q}\right) = \widehat{b}_F\left(u^*, \widehat{v}^{**}, \widehat{q}\right) \end{aligned}$$

By passing to the limit in the system e', we obtain

$$\begin{cases} \widehat{a}_F\left(u^*, \widehat{v}^{**}, \widehat{w}\right) + \widehat{b}_F\left(u^*, \widehat{w}, \widehat{p}^{**}\right) = \displaystyle\sum_{i=1}^{3} \int_{\Gamma_0} \widehat{\lambda}_i^* \widehat{w}_i \, d\widehat{\sigma}, & \forall \widehat{w} \in \widehat{W} \\ \widehat{b}_F\left(u^*, \widehat{v}^{**}, \widehat{q}\right) = 0, & \forall \widehat{q} \in \widehat{Q} \end{cases}$$

From the Proposition 4, we know that the above system has a unique solution, so $\widehat{v}^{**} = \widehat{v}^*$ and $\widehat{p}^{**} = \widehat{p}^*$.

Classically (see [8, vol. 4, chap. VI, Prop. 7, p. 1114]), we obtain that the whole sequence $\left\{\widehat{v}^k\right\}_{k\in\mathbb{N}}$ is weakly convergent to $\widehat{v}^*$ in $\widehat{W}$ and the whole sequence $\left\{\widehat{p}^k\right\}_{k\in\mathbb{N}}$ is weakly convergent to $\widehat{p}^*$ in $\widehat{Q}$.

V) We have:

the application mapping $\widehat{W}$ onto $\left(L^2\left(\Gamma_0\right)\right)^3$

$$\widehat{w} \to \widehat{w}_{|\Gamma_0}$$

is linear and strong continuous,

the application mapping $\left(L^2\left(\Gamma_0\right)\right)^3$ onto $\mathbb{R}$

$$\mu \to \|\mu\|_{0,\Gamma_0}$$

is convex and strong continuous,

the application mapping $\mathbb{R}$ onto $\mathbb{R}$

$$t \to \frac{1}{2}t^2$$

is convex and continuous.

From the elementary properties of the composed functions, we obtain that the application mapping $\widehat{W}$ onto $\mathbb{R}$

$$\widehat{w} \to \frac{1}{2}\left\|\widehat{w}_{|\Gamma_0}\right\|_{0,\Gamma_0}^2$$

is convex and strong continuous. It follows classically that it is weak sequentially lower semi-continuous, so

$$\frac{1}{2}\left\|\widehat{v}^*_{|\Gamma_0}\right\|_{0,\Gamma_0}^2 \le \liminf_{k\to\infty} \frac{1}{2}\left\|\widehat{v}^k_{|\Gamma_0}\right\|_{0,\Gamma_0}^2$$

According to (6.3) and (6.4), the control $\widehat{\lambda}^*$ is optimal and $\frac{1}{2}\left\|\widehat{v}^*_{|\Gamma_0}\right\|_{0,\Gamma_0}^2$ is the optimal value of the cost function. $\square$

Remark. The etaps of the above proof are standard. Related results, but not including the fluid-structure interaction problems, may be founded in [9], [10] and [14].

Remark. Coupling the fluid-structure equations using the Neumann boundary control and Dirichlet boundary observation on the contact surface was employed in [11].

Remark. An open problem is to find additional conditions for the control constraint $\widehat{\lambda} \in \widehat{K}$ in order to obtain zero for the optimal value of the cost function, i.e. $\widehat{v}^*_{|\Gamma_0} = 0$.

Conclusions

The mathematical model used in this paper permits to solve the coupled fluid-cable interaction problem via partitioned procedures, i.e. we can use the well established theories and numerical procedures for solving separately the fluid and the cable equations.

The control $\widehat{\lambda}$ could be considered as the "mortar" which couples the fluid equations with the cable equations. The Mortar Method was introduced in [5].

Using the arbitrary lagrangian eulerian coordinates, we have transformed a free boundary problem in a optimal control problem. Consequently, we have studied our problem in Sobolev spaces which are more attractive than working with shape topologies.

Other positive consequence, from the numerical point of view this time, is the following: we can use a fixed mesh for solving the fluid equations by the Finite Element Method.

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