Algorithm for solving fluid-structure interaction problem on a global moving mesh

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Abstract. We present a monolithic semi-implicit algorithm for solving fluid-structure interaction problem at small structural displacements. The algorithm uses one global mesh for the fluid-structure domain obtained by gluing the fluid and structure meshes which are matching on the interface. The continuity of velocity at the interface is automatically satisfied and the continuity of stress does not appear explicitly in the global weak form due to the action and reaction principle. At each time step, we have to solve a monolithic system of unknowns velocity and pressure defined on the global fluid-structure domain. Numerical results are presented.

Keywords: fluid-structure interaction; monolithic approach; semi-implicit algorithm

1. Introduction

We propose in this paper a monolithic semi-implicit algorithm to solve fluid-structure interaction problem at small structural displacements. The fluid is assumed to be governed by Navier-Stokes equations and the structure is governed by linear elasticity equations. Usually, the structure equations are written using Lagrangian coordinates and the fluid equations are written in the Eulerian framework. But the boundary conditions at the fluid-structure interface are easily implemented when the structure equations as well as the fluid equations are written in the Eulerian domain. We have introduced a global fluid-structure mesh obtained by gluing the fluid and structure meshes which are matching on the interface. An approach using a global fluid-structure mesh has been presented in Dunne (2006) for immersed interface problems, but the global mesh is fixed and the position of the fluid-structure interface is determined at each time by the level set method.

We have defined the fluid-structure pressure by extending the fluid pressure on the structure domain. At each time step, we have to solve a linear system of unknowns velocity and pressure defined on the global fluid-structure domain. We point out that the global mesh is moving in time contrary to Dunne (2006). Since the velocity is continuous all over the global domain, the continuity of the velocity at the interface is automatically satisfied. Moreover, the test functions are globally continuous and using the action and reaction principle, the continuity of stress at the interface does not appear explicitly in the global weak formulation.

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Monolithic implicit algorithms for solving nonlinear fluid-structure interaction were introduced in (Dunne 2006, Heil *et al*. 2008, Hubner *et al*. 2004). In Hron and Turek (2006), the algorithm use the same time discretization scheme for the fluid and structure equations where the global nonlinear algebraic system was solved by Newton method and in Heil *et al*. (2008) the author has compared the CPU time between a monolithic algorithm and an implicit partitioned procedures one. The paper Hubner *et al*. (2004) deals with space-time finite elements for solving the monolithic algorithm. Space-time finite element techniques for fluid-structure interactions problems are presented in Tezduyar *et al*. (2006).

The term "semi-implicit" used here means that the interface between the fluid and the structure is computed in explicit way while the fluid-structure velocity and pressure in implicit way. Semi-implicit algorithm based on Chorin-Temam projection method was introduced in Fernàndez *et al*. (2007), where the main idea was to decouple the fluid velocity computation from the strong coupled fluid-structure system which involves only the pressure and the structure velocity as unknowns. Other semi-implicit partitioned procedures algorithms have been used in (Badia *et al*. 2008a,b, Murea 2008, Murea and Sy 2009, Sy and Murea 2008, Quaini and Quarteroni 2007) for fluid-structure interaction problem. The advantage in general to use the semi-implicit algorithm instead of the implicit one is that, with the semi-implicit algorithm, the geometric non-linearity disappears. Semi-implicit monolithic algorithms based on algebraic approaches can be found in (Badia *et al*. 2008a,b, Quaini and Quarteroni 2007).

The method that we use in this paper is different from the methods proposed in the papers cited before:

- the continuity of stress is canceled from weak formulation of the coupling system and the continuity of velocity is automatically satisfied, contrary to (Badia *et al.* 2008a,b, Quaini and Quarteroni 2007) where the load from the fluid acting on the structure have to be computed;
- the characteristic functions used here give the freedom to choose the fluid and structure time discretization schemes independently, contrary to Hron and Turek (2006) where the same time discretization schemes is employed all over the domain; we employ a first order time-discretization scheme for the fluid equations and a second order time-discretization scheme for the structure equations, without demonstrating that our algorithm is more accurate than an algorithm which uses a first order time discretization for both fluid and structure problems;
- the fluid-structure interface is computed in explicit way allows to know at each time step the position of the interface, contrary to Dunne (2006) where the position of the interface is determined by the level set method.

2. Problem setting

We are interested by fluid-structure interaction problem in two dimensions. Let us denote by Ω_0^S the undeformed structure domain and we suppose that its boundary $\partial\Omega_0^s$ admits decomposition $\partial\Omega_0^S = \Gamma_D \cup \Gamma_N \cup \Gamma_0$, with $\Gamma_D = [AB] \cup [CD]$ and $\Gamma_N = [AD]$ (see Fig. 1 at the left). We denote by Ω_0^F the initial fluid domain bounded by: Σ_1 the inflow section, Σ_2 the bottom boundary, Σ_3 the outflow section and Γ_0 the top boundary. The boundary Γ_0 is common of both domains and it represents the fluid-structure interface. We have assumed that Ω_0^F and Ω_0^S are opened sets and we denote by $\Omega_0 = \Omega_0^S \cup \Omega_0^F \cup \Gamma_0$ the initial fluid-structure domain. We have added Γ_0 in order to get Ω_0 a connected domain.

Under the action of the fluid stress, the structure will be deformed. At the time instant *t*, the fluid occupies the domain Ω_t^F bounded by the moving interface Γ_t and the rigid boundary

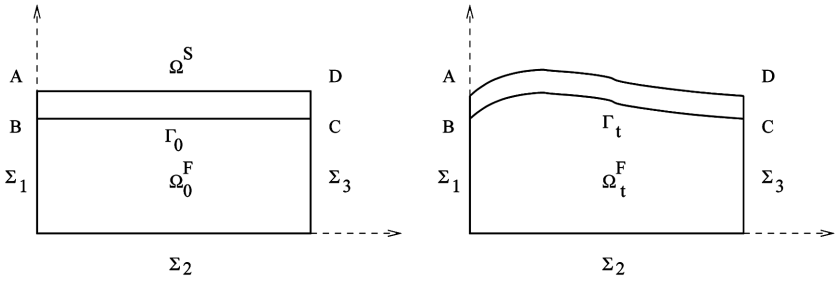

Fig. 1 Initial (left) and intermediate (right) geometrical configuration

$\Sigma = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and the structure occupies the domain Ω_t^S bounded by fluid-structure interface Γ_t, the rigid boundary Γ_D and the top boundary Γ_N (see Fig. 1 at the right). The fluid-structure domain at time instant t will be denoted by $\Omega_t = \Omega_t^S \cup \Omega_t^F \cup \Gamma_t$.

We suppose that the fluid is governed by Navier-Stokes equations and the structure is governed by linear elasticity equations. The coupling between the fluid and the structure is realized through two boundary conditions at the interface, namely, the continuity of the velocity and the equality of the stress. At each time $t \in [0, T]$, we are interested to know: the fluid velocity $\mathbf{v}^F(t) = (v_1^F(t), v_2^F(t))^T : \Omega_t^F \to \mathbb{R}^2$, the fluid pressure $p^F(t) : \Omega_t^F \to \mathbb{R}$ and the structure displacement $\mathbf{u}^S(t) = (u_1^S(t), u_2^S(t))^T : \Omega_0^S \to \mathbb{R}^2$.

The Arbitrary Eulerian Lagrangian (ALE) transformation is well adopted to solve fluid equations in moving domain (see Quarteroni and Formaggia (2004)). Let $\hat{\Omega}^F$ be a reference fluid domain and let $\mathscr{A}_t$, $t \in [0, \mathrm{T}]$ be a family of transformations such that: $\mathscr{A}_t(\hat{\mathbf{x}}) = \hat{\mathbf{x}}$ for all $\hat{\mathbf{x}} \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3$ and $\mathscr{A}_t(\hat{\Omega}^F) = \Omega_t^F$, where $\hat{\mathbf{x}} = (\hat{x}_1, \hat{x}_2)^T \in \hat{\Omega}^F$ are the ALE coordinates and $\mathbf{x} = (x_1, x_2)^T = \mathscr{A}_t(\hat{\mathbf{x}})$ the Eulerian coordinates.

Let $\mathbf{v}^F$ be the fluid velocity in the Eulerian coordinates. We denote by $\hat{\mathbf{v}}^F : \hat{\Omega}^F \to \mathbb{R}^2$ the corresponding function in the ALE coordinates, which is defined by

$$\hat{\mathbf{v}}^F(\hat{\mathbf{x}}, t) = \mathbf{v}^F(\mathscr{A}_t(\hat{\mathbf{x}}), t) = \mathbf{v}^F(\mathbf{x}, t)$$

We denote the mesh velocity by

$$\vartheta(\mathbf{x}, t) = \frac{\partial \mathscr{A}_t}{\partial t}(\hat{\mathbf{x}}) = \frac{\partial \mathscr{A}_t}{\partial t}(\mathscr{A}_t^{-1}(\mathbf{x}))$$

and the ALE time derivative of the fluid velocity by

$$\left.\frac{\partial \mathbf{v}^F}{\partial t}\right|_{\hat{\mathrm{x}}} (\mathbf{x}, t) = \frac{\partial \hat{\mathbf{v}}^F}{\partial t}(\hat{\mathbf{x}}, t)$$

We assume that the fluid-structure interaction is governed by the following equations

Navier Stokes equations

$$\rho^F\left(\left.\frac{\partial \mathbf{v}^F}{\partial t}\right|_{\hat{\mathrm{x}}} + ((\mathbf{v}^F - \vartheta) \cdot \nabla)\mathbf{v}^F\right) - 2\mu^F \nabla \cdot \varepsilon(\mathbf{v}^F) + \nabla p^F = \mathbf{f}^F, \quad \text{in } \Omega_t^F \times (0, T] \tag{1}$$

$$\nabla \cdot \mathbf{v}^F = 0, \text{ in } \Omega_t^F \times (0, T] \tag{2}$$

$$\sigma^F(\mathbf{v}^F, p^F)\cdot \mathbf{n}^F = \mathbf{h}_{in}, \quad \text{on } \Sigma_1 \times (0,T] \tag{3}$$

$$\sigma^F(\mathbf{v}^F, p^F)\cdot \mathbf{n}^F = \mathbf{h}_{out}, \quad \text{on } \Sigma_3 \times (0,T] \tag{4}$$

$$\mathbf{v}^F = 0, \quad \text{on } \Sigma_2 \times (0,T] \tag{5}$$

$$\mathbf{v}^F(\mathbf{X},0) = \mathbf{v}^0(\mathbf{X}), \text{ in } \Omega_t^F \tag{6}$$

Linear elasticity equations

$$\rho^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} - \nabla\cdot\sigma^S(\mathbf{u}^S) = \mathbf{f}^S, \quad \text{in } \Omega_0^S \times (0,T] \tag{7}$$

$$\mathbf{u}^S = 0, \quad \text{on } \Gamma_D \times (0,T] \tag{8}$$

$$\sigma^S(\mathbf{u}^S)\mathbf{n}^S = 0, \quad \text{on } \Gamma_N \times (0,T] \tag{9}$$

$$\mathbf{u}^S(\mathbf{X},0) = \mathbf{u}^0(\mathbf{X}), \quad \text{in } \Omega_0^S \tag{10}$$

Coupling conditions

$$\mathbf{v}^F(\mathbf{X}+\mathbf{u}^S(\mathbf{X},t),t) = \frac{\partial \mathbf{u}^S}{\partial t}(\mathbf{X},t), \quad \text{on } \Gamma_0 \times (0,T] \tag{11}$$

$$(\sigma^F\mathbf{n}^F)_{(\mathbf{X}+\mathbf{u}^S(\mathbf{X},t),t)} = -(\sigma^S\mathbf{n}^S)_{(\mathbf{X},t)}, \quad \text{on } \Gamma_0 \times (0,T] \tag{12}$$

where the rate of strain tensor is denoted by $\varepsilon(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v} + (\nabla\mathbf{v})^T)$ and the stress tensors are

$$\sigma^F(\mathbf{v}^F, p^F) = -p^F\mathbb{I}_2 + 2\mu^F\varepsilon(\mathbf{v}^F), \ \sigma^S(\mathbf{u}^S) = \lambda^S(\nabla\cdot\mathbf{u}^S)\mathbb{I}_2 + 2\mu^S\varepsilon(\mathbf{u}^S)$$

Moreover, $\rho^F > 0$ is the mass density of the fluid (respectively $\rho^S > 0$ the density mass of the structure), μ^F is the viscosity of the fluid (respectively μ^S, and λ^S are the Lamé's coefficients), $\mathbf{f}^F = (f_1^F, f_2^F)$ are the applied volume forces of the fluid, in general the gravity forces, (respectively $\mathbf{f}^S = (f_1^S, f_2^S)$ the applied volume forces on the structure), $\mathbf{h}_{in}$ (respectively $\mathbf{h}_{out}$) is the prescribed boundary stress on Σ_1 (respectively on Σ_3), $\mathbf{n}^S = (n_1^S, n_2^S)$ is the structure unit outward normal to Γ_0, $\mathbf{n}^F = (n_1^F, n_2^F)$ is the fluid unit outward normal to Γ_t and $\mathbb{I}_2$ is the unitary matrix.

Without risk of confusion, we use the same notation ∇ for the gradient operator with respect to Lagrangian coordinates $\mathbf{X}$ or Eulerian coordinates $\mathbf{x}$. We use the same convention for the divergence and Laplacian operators.

3. Time discretization

The time discretization of the fluid equations is based on the backward Euler scheme and a linearization of the convective term. We use a Newmark scheme of second order for the time approximation of the structure equations. Let N be the number of time step and $\Delta t = T/N$ the time step. We set $t_n = n\Delta t$ for all $n = 0,\cdots,N$ the subdivision points of $[0,T]$. We denote by $\mathbf{u}^{S,n}$, $\dot{\mathbf{u}}^{S,n}$, $\ddot{\mathbf{u}}^{S,n}$ the approximations of $\mathbf{u}^S(t_n)$, $\partial\mathbf{u}^S/\partial t(t_n)$, $\partial^2\mathbf{u}^S/\partial t^2(t_n)$ respectively, and $\mathbf{v}^{F,n}$, $p^{F,n}$ the approximations of $\mathbf{v}^F(t_n)$, $p^F(t_n)$, respectively.

We define $\vartheta^n = (\vartheta_1^n, \vartheta_2^n)^T$ the velocity of the fluid domain as solution of

$$\begin{cases} \Delta\vartheta^n = 0, \ \Omega_n^F \\ \vartheta^n = 0, \ \partial\Omega_n^F \setminus \Gamma_n \\ \vartheta^n = \mathbf{v}^{F,n}, \Gamma_n \end{cases} \tag{13}$$

For the ALE domain, we set $\hat{\Omega}^F = \Omega_n^F$ and for all $n = 0,\cdots,N-1$, we denote by $\mathscr{A}_{t_{n+1}}$ the map from $\overline{\Omega}_n^F$ to $\mathbb{R}^2$ defined by

$$\mathscr{A}_{t_{n+1}}(\hat{x}_1, \hat{x}_2) = (\hat{x}_1 + \Delta t\vartheta_1^n, \hat{x}_2 + \Delta t\vartheta_2^n)$$

We set $\Omega_{n+1}^F = \mathscr{A}_{t_{n+1}}(\Omega_n^F)$ and we define the map:

$$\mathbb{T} = \mathscr{A}_{t_n} \circ \mathscr{A}_{t_{n-1}} \cdots \circ \mathscr{A}_{t_1}$$

then, we may observe that $\Gamma_n = \mathbb{T}(\Gamma_0)$.

We define the fluid velocity $\hat{\mathbf{v}}^{F,n+1} : \Omega_t^F \to \mathbb{R}^2$ (respectively the fluid pressure $\hat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$ at time instant t_{n+1} on Ω_n^F by

$$\hat{\mathbf{v}}^{F,n+1}(\hat{\mathbf{x}}) = \mathbf{v}^{F,n+1}(\mathbf{x}), \ \hat{p}^{F,n+1}(\hat{\mathbf{x}}) = p^{F,n+1}(\mathbf{x}), \ \forall \hat{\mathbf{x}} \in \Omega_n^F, \mathbf{x} = \mathscr{A}_{t_{n+1}}(\hat{\mathbf{x}}) \in \Omega_{n+1}^F$$

Discrete fluid equations

Find $\hat{\mathbf{v}}^{F,n+1} : \Omega_n^F \to \mathbb{R}^2$ and $\hat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$ such that

$$\rho^F\left(\frac{\hat{\mathbf{v}}^{F,n+1} - \mathbf{v}^{F,n}}{\Delta t} + ((\mathbf{v}^{F,n} - \vartheta^n) \cdot \nabla)\hat{\mathbf{v}}^{F,n+1}\right)$$
$$-2\mu^F \nabla \cdot \varepsilon(\hat{\mathbf{v}}^{F,n+1}) + \nabla\hat{p}^{F,n+1} = \hat{\mathbf{f}}^{F,n+1}, \ \text{ in } \Omega_n^F \tag{14}$$

$$\nabla \cdot \hat{\mathbf{v}}^{F,n+1} = 0, \ \text{ in } \Omega_n^F \tag{15}$$

$$\sigma^F(\hat{\mathbf{v}}^{F,n+1}, \hat{p}^{F,n+1}) \cdot \mathbf{n}^F = \mathbf{h}_{in}^{n+1}, \ \text{ on } \Sigma_1 \tag{16}$$

$$\sigma^F(\hat{\mathbf{v}}^{F,n+1}, \hat{p}^{F,n+1}) \cdot \mathbf{n}^F = \mathbf{h}_{out}^{n+1}, \ \text{ on } \Sigma_3 \tag{17}$$

$$\hat{\mathbf{v}}^{F,n+1} = 0, \ \text{ on } \Sigma_2 \tag{18}$$

$$\mathbf{v}^F(\mathbf{X}, 0) = \mathbf{v}^0(\mathbf{X}), \ \text{in } \Omega_0^F \tag{19}$$

Discrete structure equations

Find $\mathbf{u}^{S,n+1}, \dot{\mathbf{u}}^{S,n+1}, \ddot{\mathbf{u}}^{S,n+1} : \Omega_0^S \to \mathbb{R}^2$ such that

$$\rho^S\ddot{\mathbf{u}}^{S,n+1} - \nabla \cdot \sigma^S(\mathbf{u}^{S,n+1}) = \mathbf{f}^{S,n+1}, \ \text{ in } \Omega_0^s \tag{20}$$

$$\mathbf{u}^{S,n+1} = 0, \ \text{ on } \Gamma_D \tag{21}$$

$$\sigma^S(\mathbf{u}^{S,n+1})\mathbf{n}^S = 0, \quad \text{on } \Gamma_N \tag{22}$$

$$\mathbf{u}^S(\mathbf{X}, 0) = \mathbf{u}^0(\mathbf{X}), \quad \text{in } \Omega_0^s \tag{23}$$

$$\dot{\mathbf{u}}^{S,n+1} = \dot{\mathbf{u}}^{S,n} + \Delta t[(1-\delta)\ddot{\mathbf{u}}^{S,n} + \delta\ddot{\mathbf{u}}^{S,n+1}] \tag{24}$$

$$\mathbf{u}^{S,n+1} = \mathbf{u}^n + \Delta t\dot{\mathbf{u}}^{S,n} + (\Delta t)^2\left[\left(\frac{1}{2}-\theta\right)\ddot{\mathbf{u}}^{S,n} + \theta\ddot{\mathbf{u}}^{n+1}\right] \tag{25}$$

For $\delta = 1/2$, the Newmark scheme is of second order in time.

Coupling conditions

$$\hat{\mathbf{v}}^{F,n+1}\circ\mathbb{T} = \dot{\mathbf{u}}^{S,n+1}, \quad \text{on } \Gamma_0 \times (0,T] \tag{26}$$

$$(\sigma^F(\hat{\mathbf{v}}^{F,n+1}, \hat{p}^{F,n+1})\mathbf{n}^F)\circ\mathbb{T} = -\sigma^S(\mathbf{u}^{S,n+1})\mathbf{n}^S, \quad \text{on } \Gamma_0 \times (0,T] \tag{27}$$

4. Weak formulation of the time discrete equations

We define the spaces of test functions $\hat{W}_n^F$ for the fluid velocity and $\hat{Q}_n^F$ for the fluid pressure as following

$$\hat{W}_n^F = \{\hat{\mathbf{w}}^F \in (H^1(\Omega_n^F))^2 \,;\, \hat{\mathbf{w}}^F = 0 \text{ on } \Sigma_2\}, \text{ and } \hat{Q}_n^F = L^2(\Omega_n^F).$$

We multiply the Eq. (14) by a test function $\hat{\mathbf{w}}^F \in \hat{W}_n^F$ and the Eq. (15) by a test function $\hat{q} \in \hat{Q}_n^F$, after integrating them over the domain Ω_n^F by part, and using the corresponding boundary conditions, we get the following discrete weak form:
Find $\hat{\mathbf{v}}^{F,n+1} \in \hat{W}_n^F$ and $\hat{p}^{F,n+1} \in \hat{Q}_n^F$ such that

$$\int_{\Omega_n^F}\rho^F\frac{\hat{\mathbf{v}}^{F,n+1}}{\Delta t}\cdot\hat{\mathbf{w}}^F + \int_{\Omega_n^F}\rho^F(((\mathbf{v}^{F,n}-\mathfrak{V}^n)\cdot\nabla)\hat{\mathbf{v}}^{F,n+1})\cdot\hat{\mathbf{w}}^F$$
$$-\int_{\Omega_n^F}(\nabla\cdot\hat{\mathbf{w}}^F)\hat{p}^{F,n+1} + \int_{\Omega_n^F}2\mu^F\varepsilon(\hat{\mathbf{v}}^{F,n+1}):\varepsilon(\hat{\mathbf{w}}^F) - \int_{\Gamma_n}(\sigma^F\mathbf{n}^F)\cdot\hat{\mathbf{w}}^F = \mathscr{L}_F(\hat{\mathbf{w}}^F), \ \forall\hat{\mathbf{w}}^F \in \hat{W}_n^F \tag{28}$$

$$\int_{\Omega_n^F}\hat{q}(\nabla\cdot\hat{\mathbf{v}}^{F,n+1}) = 0, \ \forall\hat{q} \in \hat{Q}_n^F \tag{29}$$

where

$$\mathscr{L}_F(\hat{\mathbf{w}}^F) = \int_{\Omega_n^F}\rho^F\frac{\hat{\mathbf{v}}^{F,n}}{\Delta t}\cdot\hat{\mathbf{w}}^F + \int_{\Omega_n^F}\hat{\mathbf{f}}^{n+1,F}\cdot\hat{\mathbf{w}}^F + \int_{\Sigma_1}\hat{\mathbf{h}}_{in}^{n+1}\cdot\hat{\mathbf{w}}^F + \int_{\Sigma_3}\hat{\mathbf{h}}_{out}^{n+1}\cdot\hat{\mathbf{w}}^F$$

Remark 1. The system (20), (24), (25) is a system of three vectorial unknowns and three vectorial equations, we can reduce it to a system where the only unknown is the structure velocity. For that we have to express the displacement $\mathbf{u}^{S,n+1}$ and the acceleration $\ddot{\mathbf{u}}^{S,n+1}$ in function of the velocity $\dot{\mathbf{u}}^{S,n+1}$ from (25) and (24), see Murea (2008) for the details.

We now define the space of test functions for the structure velocity

$$W^S = \{\mathbf{w}^S \in (H^1(\Omega_0^S))^2 \,;\, \mathbf{w}^S = 0 \text{ on } \Gamma_D\}$$

Let us multiply the Eq. (20) by a test function $\mathbf{w}^S \in W^S$ and we integrate it over the domain Ω_0^s by part, using the corresponding boundary conditions and following the remark (1), we get the following discrete structure weak form

Find $\dot{\mathbf{u}}^{S,n+1} \in W^S$ such that:

$$\begin{aligned} &\int_{\Omega_0^S} \frac{2\rho^S}{\Delta t}\dot{\mathbf{u}}^{S,n+1} \cdot \mathbf{w}^S + 2\theta\Delta t a_S(\dot{\mathbf{u}}^{S,n+1}, \mathbf{w}^S) \\ &-\int_{\Gamma_0} (\sigma^S \mathbf{n}^S) \cdot \mathbf{w}^S = \mathscr{L}_S(\mathbf{w}^S), \;\; \forall \mathbf{w}^S \in W^S \end{aligned} \tag{30}$$

where

$$a_S(\mathbf{u}, \mathbf{w}) = \int_{\Omega_0^S} [\lambda^S(\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{w}) + 2\mu^F \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{w})]$$

and

$$\begin{aligned} \mathscr{L}_S(\mathbf{w}^S) = &\int_{\Omega_0^S} f^S \cdot \mathbf{w}^S + \int_{\Omega_0^S} \frac{2\rho^S}{\Delta t}\dot{\mathbf{u}}^{S,n} \cdot \mathbf{w}^S + \int_{\Omega_0^S} \rho^S \ddot{\mathbf{u}}^{S,n} \cdot \mathbf{w}^S - a_S(\mathbf{u}^{S,n}, \mathbf{w}^S) \\ &-\Delta t(1-2\theta)a_S(\dot{\mathbf{u}}^{S,n}, \mathbf{w}^S) - (\Delta t)^2\left(\frac{1}{2} - 2\theta\right)a_S(\ddot{\mathbf{u}}^{S,n}, \mathbf{w}^S) \end{aligned}$$

5. Monolithic formulation for the fluid-structure equations

In this section, we will formulate the fluid-structure interaction problem as a monolithic system, for that we need to define one single vector field for the velocity and to extend the fluid pressure and fluid mesh velocity on the fluid-structure domain $\Omega_t = \Omega_t^F \cup \Omega_t^S \cup \Gamma_t$. We denote the fluid-structure velocity by $\mathbf{v} = (v_1, v_2)^T : \Omega_t \to \mathbb{R}^2$ and we define the time approximation of the fluid-structure velocity, displacement and acceleration by

$$\mathbf{v}^n : \Omega_n \to \mathbb{R}^2, \;\; \mathbf{u}^n : \Omega_n \to \mathbb{R}^2, \;\; \ddot{\mathbf{u}}^n : \Omega_n \to \mathbb{R}^2$$

The characteristic functions related to the fluid domain $\chi_{\Omega_t^F} : \overline{\Omega}_t \to \mathbb{R}$ and structure domain $\chi_{\Omega_t^S} : \overline{\Omega}_t \to \mathbb{R}$ are defined by

$$\chi_{\Omega_t^S} = \begin{cases} 1, \text{ on } \overline{\Omega}_t^S \\ 0, \text{ otherwise} \end{cases} \qquad \chi_{\Omega_t^F} = 1 - \chi_{\Omega_t^S}$$

We denote by Ω_n the time approximation space of Ω_t and we define the space of test functions $\hat{W}_n$ for the fluid-structure velocity by

$$\hat{W}_n = \{\hat{\mathbf{w}} \in (H^1(\Omega_n))^2 \,;\, \hat{\mathbf{w}} = 0 \text{ on } \Gamma_D \cup \Sigma_2\}$$

In order to write the structure equations on the intermediate domain Ω_n. Let us denote by $\phi : \Omega_0 \to \Omega_n$ the map which is defined by: $\phi(X) = X + \mathbf{u}^n,\ X \in \Omega_0$. Therefore, we have

$$\int_{\Omega_n} f(\mathrm{x}) d\mathrm{x} = \int_{\Omega_0} f(\phi(X)) \mathbf{J}_n dX$$

where $\mathbf{J}_n = \det(\mathbb{I}_2 + \mathbf{u}^n)$.

In our case, we recall that we work with the small structural displacements, therefore $\mathbf{J}_n$ can be approximate by 1. Now applying this theory to our system (30), we can approximate it by the following:

$$\begin{aligned} &\int_{\Omega_n} \chi_{\Omega_n^S} \frac{2\rho^S}{\Delta t} \hat{\mathbf{v}}^{n+1} \cdot \hat{\mathbf{w}} + 2\theta \Delta t \tilde{\mathrm{a}}_S(\hat{\mathbf{v}}^{n+1}, \hat{\mathbf{w}}) \\ &- \int_{\Gamma_n} (\sigma^S \mathbf{n}^S) \cdot \hat{\mathbf{w}} = \tilde{\mathscr{G}}_S(\hat{\mathbf{w}}),\ \forall \hat{\mathbf{w}} \in \hat{W}_n \end{aligned} \tag{31}$$

where

$$\tilde{\mathrm{a}}_S(\mathbf{u}, \mathbf{w}) = \int_{\Omega_n} \chi_{\Omega_n^S} [\lambda^S (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{w}) + 2\mu^S \varepsilon(\mathbf{u}) : \varepsilon(\mathbf{w})]$$

and

$$\tilde{\mathscr{G}}_S(\hat{\mathbf{w}}) = \int_{\Omega_n} \chi_{\Omega_n^S} f^S \cdot \hat{\mathbf{w}} + \int_{\Omega_n} \chi_{\Omega_n} \frac{2\rho^S}{\Delta t} \mathbf{v}^n \cdot \hat{\mathbf{w}} + \int_{\Omega_n} \chi_{\Omega_n^S} \rho^S \ddot{\mathbf{u}}^n \cdot \hat{\mathbf{w}}$$

$$- \tilde{\mathrm{a}}_S(\mathbf{u}^n, \hat{\mathbf{w}}) - \Delta t(1 - 2\theta) \tilde{\mathrm{a}}_S(\mathbf{v}^n, \hat{\mathbf{w}}) - (\Delta t)^2 \left(\frac{1}{2} - 2\theta\right) \tilde{\mathrm{a}}_S(\ddot{\mathbf{u}}^n, \hat{\mathbf{w}})$$

The fluid-structure mesh velocity $\tilde{\mathfrak{I}}^n = (\tilde{\mathfrak{I}}_1^n, \tilde{\mathfrak{I}}_2^n)$ is obtained as the extension of the fluid mesh velocity on whole domain Ω_n

$$\begin{cases} -\Delta \tilde{\mathfrak{I}}^n = 0, & \text{in } \Omega_n \\ \tilde{\mathfrak{I}}^n = 0 & \text{in } \partial\Omega_n \\ \tilde{\mathfrak{I}}^n = \mathbf{v}^n, & \text{on } \Gamma_n. \end{cases} \tag{32}$$

Remark 2. We can notice that Γ_n is a curve which separates Ω_n^F and Ω_n^S, therefore the problem (32) is well posed.

We introduce a global pressure $p : \Omega_t \to \mathbb{R}$, which coincides with the fluid pressure p^F on the fluid domain Ω_t^F. The value of p on the structure domain Ω_t^S has no physical signification. We will return to this point later. We denote by $\hat{p}^{n+1} : \Omega_n \to \mathbb{R}$ the time approximation of p at t_{n+1}. The space of test function for the pressure is denoted by $\hat{Q}_n = L^2(\Omega_n)$.

From (27), we have

$$\int_{\Gamma_n} (\sigma^S \mathbf{n}^S) \cdot \hat{\mathbf{w}} + \int_{\Gamma_n} (\sigma^F \mathbf{n}^F) \cdot \hat{\mathbf{w}} = 0,\ \forall \hat{\mathbf{w}} \in \hat{W}_n \tag{33}$$

Using the fluid characteristic function, we can replace (28) and (29) by

Find $\hat{\mathbf{v}}^{F,n+1} \in \hat{W}_n$ and $\hat{p}^{F,n+1} \in \hat{Q}_n$ such that

$$\int_{\Omega_n} \chi_{\Omega_n^F} \rho^F \frac{\hat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \hat{\mathbf{w}} + \int_{\Omega_n} \chi_{\Omega_n^F} \rho^F (((\mathbf{v}^n - \tilde{\mathfrak{I}}^n) \cdot \nabla) \hat{\mathbf{v}}^{n+1}) \cdot \hat{\mathbf{w}}$$
$$- \int_{\Omega_n} \chi_{\Omega_n^F} (\nabla \cdot \hat{\mathbf{w}}) \hat{p}^{n+1} + \int_{\Omega_n} \chi_{\Omega_n^F} 2\mu^F \varepsilon(\hat{\mathbf{v}}^{n+1}) : \varepsilon(\hat{\mathbf{w}})$$
$$- \int_{\Gamma_n} (\sigma^F \mathbf{n}^F) \cdot \hat{\mathbf{w}} = \tilde{\mathscr{L}}_F(\hat{\mathbf{w}}), \ \forall \hat{\mathbf{w}} \in \hat{W}_n \tag{34}$$

$$\int_{\Omega_n} \chi_{\Omega_n^F} \hat{q} (\nabla \cdot \hat{\mathbf{v}}^{n+1}) = 0, \ \forall \hat{q} \in \hat{Q}_n \tag{35}$$

where

$$\tilde{\mathscr{L}}_F(\hat{\mathbf{w}}) = \int_{\Omega_n} \chi_{\Omega_n^F} \rho^F \frac{\mathbf{v}^n}{\Delta t} \cdot \hat{\mathbf{w}} + \int_{\Omega_n} \chi_{\Omega_n^F} \hat{\mathbf{f}}^{n+1,F} \cdot \hat{\mathbf{w}} + \int_{\Sigma_1} \hat{\mathbf{h}}_{in}^{n+1} \cdot \hat{\mathbf{w}} + \int_{\Sigma_3} \hat{\mathbf{h}}_{out}^{n+1} \cdot \hat{\mathbf{w}}$$

Now using the identity (33) and summing up the Eqs. (31) and (34), we get after adding the Eq. (35) the following fluid-structure weak formulation:

Find $(\hat{\mathbf{v}}^{n+1}, \hat{p}^{n+1}) \in \hat{W}_n \times \hat{Q}_n$ such that

$$\int_{\Omega_n} \chi_{\Omega_n^F} \rho^F \frac{\hat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \hat{\mathbf{w}} + \int_{\Omega_n} \chi_{\Omega_n^F} \rho^F (((\mathbf{v}^n - \tilde{\mathfrak{I}}^n) \cdot \nabla) \hat{\mathbf{v}}^{n+1}) \cdot \hat{\mathbf{w}}$$
$$- \int_{\Omega_n} \chi_{\Omega_n^F} (\nabla \cdot \hat{\mathbf{w}}) \hat{p}^{n+1} + \int_{\Omega_n} \chi_{\Omega_n^F} 2\mu^F \varepsilon(\hat{\mathbf{v}}^{n+1}) : \varepsilon(\hat{\mathbf{w}})$$
$$+ \int_{\Omega_n} \chi_{\Omega_n^S} \frac{2\rho^S}{\Delta t} \hat{\mathbf{v}}^{n+1} \cdot \hat{\mathbf{w}} + 2\theta \Delta t \tilde{\mathrm{a}}_S(\hat{\mathbf{v}}^{n+1}, \hat{\mathbf{w}})$$
$$= \tilde{\mathscr{L}}_F(\hat{\mathbf{w}}) + \tilde{\mathscr{L}}_S(\hat{\mathbf{w}}), \ \forall \hat{\mathbf{w}} \in \hat{W}_n \tag{36}$$

$$\int_{\Omega_n} \chi_{\Omega_n^F} \hat{q} (\nabla \cdot \hat{\mathbf{v}}^{n+1}) = 0, \ \forall \hat{q} \in \hat{Q}_n \tag{37}$$

The fluid-structure displacement at time instant t_{n+1} on Ω_n : $\hat{\mathbf{u}}^{n+1} : \Omega_n \to \mathbb{R}^2$ can be computed from (24), (25) and from the remark (1), as following

$$\hat{\mathbf{u}}^{n+1} = \mathbf{u}^n + (\Delta t)^2 \left(\frac{1}{2} - 2\theta\right) \ddot{\mathbf{u}}^n + \Delta t (1 - 2\theta) \mathbf{v}^n + 2\theta \Delta t \hat{\mathbf{v}}^{n+1} \tag{38}$$

The fluid-structure acceleration $\hat{\ddot{\mathbf{u}}}^{n+1} : \Omega_n \to \mathbb{R}^2$ is obtained by combining (24), (25)

$$\hat{\ddot{\mathbf{u}}}^{n+1} = \frac{2}{\Delta t} (\hat{\mathbf{v}}^{n+1} - \mathbf{v}^n) - \ddot{\mathbf{u}}^n \tag{39}$$

We define the map $\mathbb{T}_n : \overline{\Omega}_n \to \mathbb{R}^2$ by:

$$\mathbb{T}_n(\hat{\mathbf{x}}) = \hat{\mathbf{x}} + \Delta t \tilde{\mathfrak{I}}^n(\hat{\mathbf{x}}) \chi_{\Omega_n^F}(\hat{\mathbf{x}}) + (\hat{\mathbf{u}}^{n+1}(\hat{\mathbf{x}}) - \mathbf{u}^n(\hat{\mathbf{x}})) \chi_{\Omega_n^S}(\hat{\mathbf{x}}) \tag{40}$$

We set

$$\Omega_{n+1} = \mathbb{T}_n(\Omega_n)$$

and we can observe that

$$\Gamma_{n+1} = \mathbb{T}_n(\Gamma_n)$$

We now define the fluid-structure velocity and pressure defined on the current time domain, $\mathbf{v}^{n+1}$: $\Omega_{n+1} \to \mathbb{R}^2$ and $p^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ by

$$\mathbf{v}^{n+1}(\mathbf{x}) = \hat{\mathbf{v}}^{n+1}(\hat{\mathbf{x}}),\ p^{n+1}(\mathbf{x}) = \hat{p}^{n+1}(\hat{\mathbf{x}}),\ \forall \hat{\mathbf{x}} \in \Omega_n \text{ and } \mathbf{x} = \mathbb{T}_n(\hat{\mathbf{x}})$$

In the similar way, we define the fluid-structure displacements and acceleration at time instant t_{n+1} on Ω_{n+1}:$\mathbf{u}^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ and $\ddot{\mathbf{u}}^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ by

$$\mathbf{u}^{n+1}(\mathbf{x}) = \hat{\mathbf{v}}^{n+1}(\hat{\mathbf{x}}),\ \ddot{\mathbf{u}}^{n+1}(\mathbf{x}) = \hat{\ddot{\mathbf{u}}}^{n+1}(\hat{\mathbf{x}}),\ \forall \hat{\mathbf{x}} \in \Omega_n \text{ and } \mathbf{x} = \mathbb{T}_n(\hat{\mathbf{x}})$$

In order to solve the linear system obtained from (36), (37) after the finite elements discretization, we choose the direct solver **LU**. The linear system has form

$$\begin{bmatrix} A & B^T & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} V \\ P^F \\ P^S \end{bmatrix} = \begin{bmatrix} \mathscr{L} \\ 0 \\ 0 \end{bmatrix} \tag{41}$$

where $\mathscr{L} = \tilde{\mathscr{L}}_F(\phi_i) + \tilde{\mathscr{L}}_S(\phi_i)$ and the matrices are defined by

$$A_{i,j} = \int_{\Omega_n} \chi_{\Omega_n^F} \rho^F \frac{\phi_j}{\Delta t} \cdot \phi_i + \int_{\Omega_n} \chi_{\Omega_n^F} \rho^F (((\mathbf{v}^n - \tilde{\mathfrak{I}}^n) \cdot \nabla)\phi_j) \cdot \phi_i$$
$$+ \int_{\Omega_n} \chi_{\Omega_n^F} 2\mu^F \varepsilon(\phi_j) : \varepsilon(\phi_i) + \int_{\Omega_n} \chi_{\Omega_n^S} \frac{2\rho^S}{\Delta t} \phi_j \cdot \phi_i + 2\theta \Delta t \tilde{\mathrm{a}}_S(\phi_j, \phi_i)$$

and

$$B_{i,j} = -\int_{\Omega_n} \chi_{\Omega_n^F} (\nabla \cdot \phi_j) \pi_i$$

with ϕ_i (respectively π_i) the velocity (respectively the pressure) finite element shape functions. The unknowns are: V the velocity vector, P^F the pressure vector on the mesh nodes in the fluid domain **including** the fluid-structure interface, P^S the pressure vector on the mesh nodes in the structure domain **excluding** the fluid-structure interface.

We can remark that this system has non unique solution, because the pressure P^S can take any value. To get round this difficulty we have added to the system (36), (37) the term $\varepsilon \int_{\Omega_n} \hat{p}^{n+1} \hat{q}$ and the global matrix becomes:

$$\begin{bmatrix} A & B^T & 0 \\ B & \varepsilon M^F & 0 \\ 0 & 0 & \varepsilon M^S \end{bmatrix} \begin{bmatrix} V \\ P^F \\ P^S \end{bmatrix} = \begin{bmatrix} \mathscr{L} \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

where

$$\begin{bmatrix} \varepsilon M^F & 0 \\ 0 & \varepsilon M^S \end{bmatrix} = \left(\varepsilon \int_{\Omega_n} \pi_j \pi_i \right)_{1 \le i,j \le nv}$$

with nv the number of vertices of the fluid-structure mesh.

The influence of ε the regularization parameter will be analyzed in the section devoted to numerical results.

Remark 3. We approach the global pressure by $\mathbb{P}_1$ finite element functions, which are globally continuous.

From (42), we obtain that $P^S = 0$. We insist on the fact that P^S represents the pressure on the mesh nodes in the structure domain, **excluding** the fluid-structure interface, so the computed pressure is not zero on the interface.

Algorithm

We have employed the following algorithm to implement numerically the linear system (36), (37).

We assume that we know Ω_n, $\mathbf{v}^n$, $\mathbf{u}^n$, $\ddot{\mathbf{u}}^n$.

Step 1: Compute $\tilde{\mathfrak{I}}^n$ from (32).

Step 2: Solve the linear system (36), (37) by LU on the mesh $\mathcal{T}_n$ of Ω_n. Get the fluid-structure velocity $\hat{\mathbf{v}}^{n+1}$ and pressure $\hat{p}^{n+1}$.

Step 3: Compute the fluid-structure displacement $\hat{\mathbf{u}}^{n+1}$ from (38) and acceleration $\hat{\ddot{\mathbf{u}}}^{n+1}$ from (39).

Step 4: Build mesh $\mathcal{T}_{n+1}$ as image of $\mathcal{T}_n$ by the map (40) and save the mesh $\mathcal{T}_{n+1}$, the velocity $\mathbf{v}^{n+1}$, the pressure p^{n+1}, the displacement $\mathbf{u}^{n+1}$ and the acceleration $\ddot{\mathbf{u}}^{n+1}$.

6. Numerical results

We are motivated by fluid-structure interaction problems with application to haemodynamics. The fluid is the blood and the structure represents the artery. But before simulating real three dimensional applications, it is important to test our method on an academic case.

Physical parameters

We consider the following data for the numerical computations: the length of the fluid domain is $L = 6\ cm$ and its height $H = 1\ cm$. The viscosity of the fluid was fixed to be $\mu = 0.035(g/cm{\cdot}s)$, its density $\rho^F = 1(g/cm^3)$ and the volume force in fluid is $\mathbf{f}^F = (0, 0)^T$. The prescribed boundary stress at the outflow is $\mathbf{h}_{out}(x, t) = (0, 0)^T$ and at the inflow is:

$$\mathbf{h}_{in}(x, t) = \begin{cases} (10^3(1 - cos(2\pi t/0.025)), 0)^T, & \text{if } x \in \Sigma_1, 0 \le t \le 0.025 \\ (0, 0)^T, & \text{if } x \in \Sigma_1, 0.025 \le t \le T. \end{cases}$$

The thickness of the elastic wall is $h^S = 0.1\ cm$, the Young modulus $E = 3{\cdot}10^6(g/cm{\cdot}s^2)$, the Poisson ratio $\nu = 0.3$, the mass density $\rho^S = 1.1(g/cm^3)$ and the volume force is $\mathbf{f}^S = (0, 0)^T$. The Lamé's coefficients are computed by the formulas

$$\lambda^S = \frac{\nu^S E}{(1 - 2\nu^S)(1 + \nu^S)}, \quad \mu^S = \frac{E}{2(1 + \nu^S)}$$

Numerical parameters

The computation has been made on a computer with two processors of 2.20 GHz frequency. The

numerical tests have been performed using FreeFem++ (see Hecht *et al.*). We have chosen the following parameters: $\delta = 0.5$ and $\theta = 0.3$ for the Newmark scheme, $\Delta t = 0.001\ s$ the time step and $N = 100$ the number of time step. We denote by m the number of eigenfunctions when the structure problem is solved by modal decomposition method in the case of partitioned procedures algorithm.

CPU time for monolithic algorithm

The number of segments on the fluid-structure interface is denoted by *nsFS*. We have used for the fluid-structure a reference mesh of number of triangles $nt = 2426$ and number of vertices $nv = 1305$. For the finite element approximation of the fluid-structure velocity, we have used the triangular finite element $\mathbb{P}_1 + bubble$ and we have employed for the pressure the finite element $\mathbb{P}_1$. For the characteristic functions, the triangular finite element $\mathbb{P}_0$ was used on each triangle. The linear fluid-structure system is solved using **LU** solver where the continuity of velocity is automatically satisfied and the continuity of stress does not appear in the global weak form. We have summarized in the Table 1 the CPU time for monolithic algorithm when different values of segments on the interface have been considered: $nsFS = 80$, $nsFS = 100$ and $nsFS = 120$. The degree of freedoms computed in the Table 1 are obtained from the formulas: $Dof = 2(nv + nt) + nv = 3nv + 2nt$, where $2(nv + nt)$ is considered for velocity and nv for the pressure.

Influence of the regularization parameter

In order to analyze the influence of ε the regularization parameter in (42), we introduce the following norm

$$\|\mathbf{v}(\varepsilon)\| = \sqrt{\Delta t \sum_{n=1}^{N} \int_{\Omega_n} (\mathbf{v}^n)^2 dx}$$

Let us consider the solution obtained for $\varepsilon = 10^{-6}$ as the reference solution $\bar{\mathbf{v}} = \mathbf{v}(10^{-6})$. We have used $N = 20$ the number of time steps, $\Delta t = 0.001$ the time step and $nsFS = 120$ the number of segments on the interface.

We conclude that the introduction of the regularization parameter does not change significantly the computed solution.

Table 1 Monolithic (mono) algorithm for three different values of *nsFS*

nsFS	*nt*	*nv*	global Dof	CPU_{mono}
80	2426	1305	8767	3 mn 46 s
100	3916	2070	14042	6 mn 9 s
120	5039	2649	18016	8 mn 12 s

Table 2 Influence of the regularization parameter

ε	10^{-5}	10^{-7}	10^{-8}
$\|\mathbf{v}(\varepsilon) - \bar{\mathbf{v}}\|$	0.000592709	5.92803e-05	6.52085e-05
$\frac{\|\mathbf{v}(\varepsilon) - \bar{\mathbf{v}}\|}{\|\bar{\mathbf{v}}\|}$	0.000359369	3.59426e-05	3.95369e-05

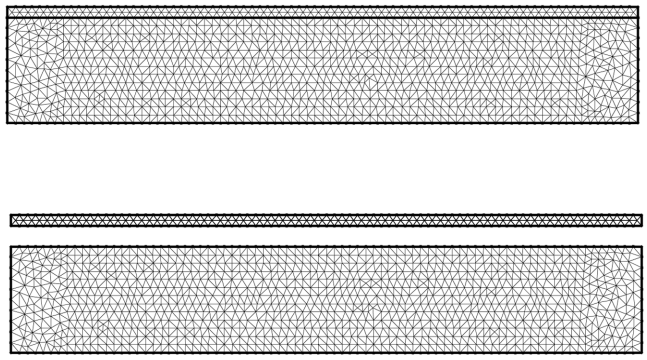

Fig. 2 Global fluid-structure mesh (top), the structure and fluid meshes (bottom)

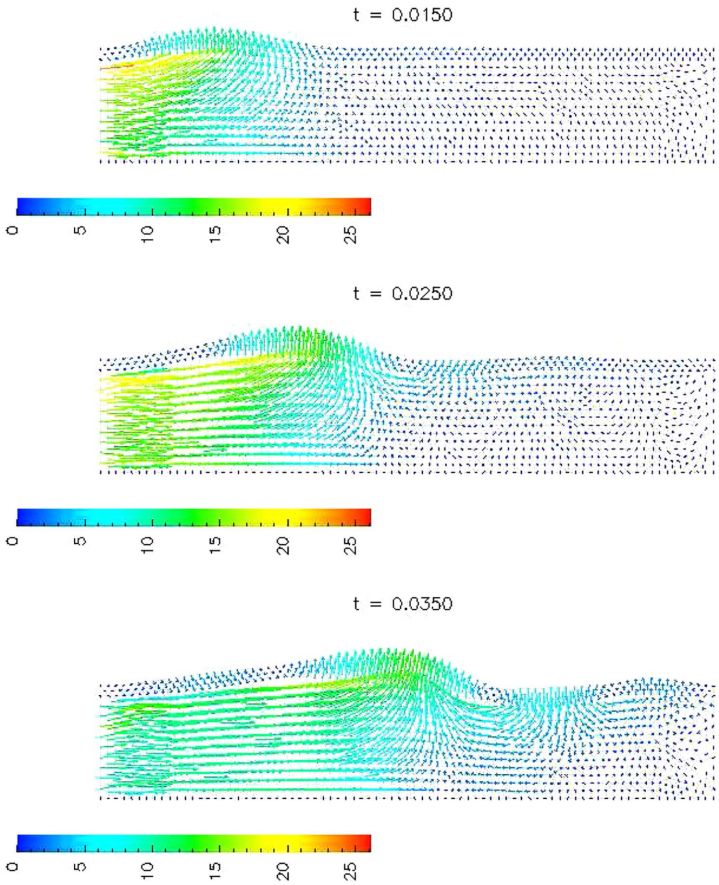

Fig. 3 Fluid-structure velocities [*cm*/*s*] at time instant $t = 0.015$ (top) , $t = 0.025$ (middle), $t = 0.035$ (bottom)

Fluid-structure mesh

The system (36), (37) is a linear system of unknowns velocity and pressure defined on the whole fluid-structure domain. The mesh $\mathcal{T}_h$ of the fluid-structure domain Ω_n is obtained by gluing the fluid and structure meshes which are compatible at the interface. The numbers of vertices and

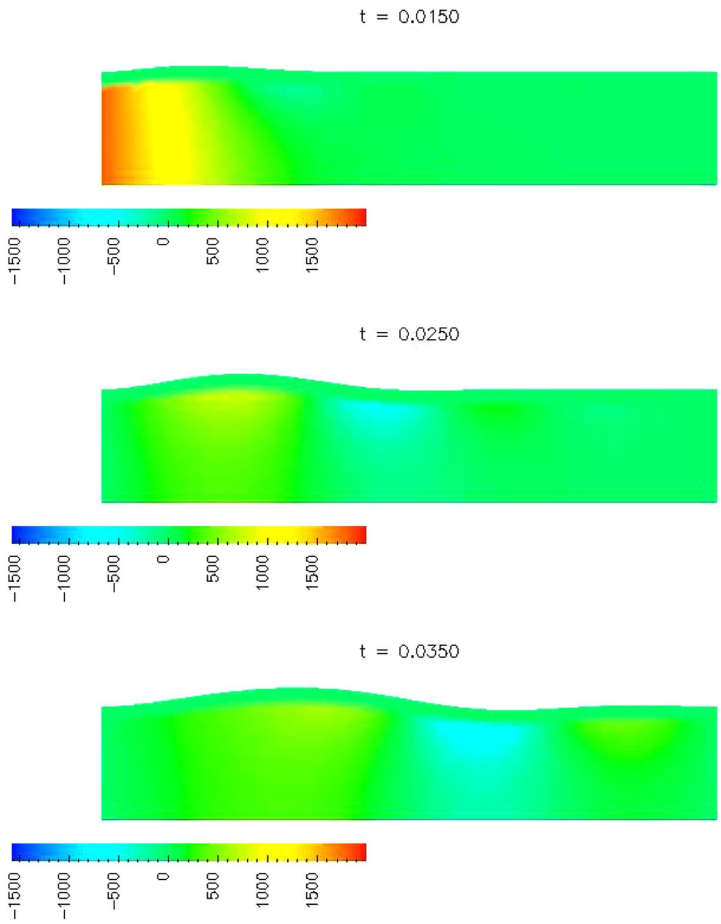

Fig. 4 Fluid-structure pressure [$dynes/s^2$] at time instant $t = 0.015$ (top), $t = 0.025$ (middle), $t = 0.035$ (bottom)

triangles of the global mesh are given by the formula

$$nv = nvF + nvS - (nsFS + 1) \quad nt = ntF + ntS$$

We plotted the global fluid-structure mesh and separately the fluid and structure meshes, see Fig. 2.

Behavior of the solutions
Fluid-structure velocities and pressure at three time instants are plotted in Figs. 3 and 4.

7. Conclusions

A monolithic semi-implicit algorithm for fluid-structure interaction problem at small structural displacement was presented. This algorithm use a global fluid-structure mesh obtained by gluing the fluid and structure meshes which are matching at the interface. The characteristic functions were

introduced in order to be able to choose independently the time discretization schemes of the fluid and structure. The continuity of velocity at the interface is automatically satisfied by using one velocity vector field and the continuity of stress is canceled from the weak form. Therefore the fluid-structure problem was written as a linear system of unknowns velocity and pressure. We used the **LU** solver in order to solve numerically the system. With this monolithic approach the CPU time is considerably small.

In the future works, we will replace the linear structure model by the nonlinear Saint-Venant Kirchhoff model.

References

Badia, S., Quaini, A. and Quarteroni, A. (2008a), "Splitting methods based on algebraic factorization for fluid-structure interaction", *SIAM J. Sci. Comput*., **30**(4), 1778-1805.

Badia, S., Quaini, A. and Quarteroni, A. (2008b), "Modular vs. non-modular preconditioners for fluidstructure systems with large added-mass effect", *Comput. Method. Appl. M*., **197**(49-50), 4216-4232.

Dunne, T. (2006),"An Eulerian approach to fluid-structure interaction and goal-oriented mesh adaptation", *Int. J. Num. Meth. Fl*., **51**(9-10), 1017-1039.

Fernández, M.A., Gerbeau, J.F. and Grandmont, C. (2007), "A projection semi-implicit scheme for the coupling of an elastic structure with an incompressible fluid", *Int. J. Numer. Meth. Eng*., **69**(4), 794-821.

Hecht, F., Pironneau, O., Le Hyaric, A. and Ohtsuka, K. (2005), FreeFem++: http://www.freefem++.org/ff++.

Hron, J. and Turek, S. (2006), *A monolithic FEM/multigrid solver for an ALE formulation of fluidstructure interaction with application in biomechanics*, In Fluid-Structure Interaction, Lect. Notes. Comput. Sci. Eng. 53, 146-170, Springer, Berlin.

Heil, M., Hazed, A.L. and Boyle, J. (2008),"Solvers for large-displacement fluid-structure interaction problems: segregated versus monolithic approaches", *Comput. Mech*., **193**(1-2), 91-101.

Hubner, B., Walhorn, E. and Dinkler, D. (2004), "A monolithic approach to fluid-structure interaction using space-time finite elements", *Comput. Meth. Appl*., **193**(23-26), 2087-2104.

Murea, C.M. (2008), "A semi-implicit algorithm based on the Augmented Lagrangian Method for fluid-structure interaction", Num. Math. Adv. App, *Proceedings of the ENUMATH 2007, the 7th European Conference on Numerical Mathematics and Advanced Applications*, Graz, Austria, September 2007, Springer, 555-562.

Murea, C.M. and Sy, S. (2009), "A fast method for solving fluid-structure interaction problem numerically", *Int. J. Numer. Meth. Fl*, **60**(10), 1149-1172.

Sy, S. and Murea, C.M. (2008), "A stable time advancing scheme for solving fluid-structure interaction problem at small structural displacements", *Comput. Meth. Appl. Mech.*, **198**(2), 210-222.

Tezduyar, T., Sathe, S., Keedy, R. and Stein, K. (2006), "Space-time finite element techniques for computation of fluidstructure interactions", *Comput. Meth. Appl. Mech*., **195**(17-18), 2002-2027.

Quaini, A. and Quarteroni, A. (2007), "A semi-implicit approach for fluid-structure interaction based on an algebraic fractional step method", *Math. Mod. Meth. Appl. S*., **17**(6), 957-983.

Quarteroni, A. and Formaggia, L. (2004), *Mathematical modelling and numerical simulation of the cardiovascular system*, (Ed. P.G. Ciarlet), Handbook of numerical analysis, vol. XII, North-Holland, Amsterdam, 3-127.