Numerical approximation of a pulsatile flow through an elastic pipe

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Content

Approximation of the structure. Natural frequencies and normal mode shapes. Newmark method

Approximations of the unsteady Stokes equations in a moving domain. ALE and time discretization. Mixed Finite Element

Approximation of the coupled fluid-structure equations. Identification of the stresses on the interface using the Least Squares Method. Coupled fluid-structure algorithm by BFGS Method

Numerical results

Strong equations of the structure

Find the transverse displacement $u : [0, L] \times [0, T] \to \mathbb{R}$ such that

$$\rho^s h \frac{\partial^2 u}{\partial t^2}(x_1, t) + \frac{E h^3}{12(1-\nu^2)} \frac{\partial^4 u}{\partial x_1^4}(x_1, t) = \eta(x_1, t),$$

$$u(0, t) = 0, \ \frac{\partial u}{\partial x_1}(0, t) = 0, \quad t \in (0, T)$$

$$u(L, t) = 0, \ \frac{\partial u}{\partial x_1}(L, t) = 0, \quad t \in (0, T)$$

$$u(x_1, 0) = u^0(x_1), \quad x_1 \in (0, L)$$

$$\frac{\partial u}{\partial t}(x_1, 0) = \dot{u}^0(x_1), \quad x_1 \in (0, L)$$

Natural frequencies and normal mode shapes

We denote by $0 < s_0 < \cdots < s_i < \ldots$ the solutions of the equation

$$\cos(s)\cosh(s) = 1, \quad s > 0.$$

We set $a_i = s_i/L$. For each $i \in \mathbb{N}$ there exists an unique normal mode shape $\phi_i \in \mathcal{C}^4([0, L])$ such that

$$\begin{aligned}
&\phi_i''''(x_1) = (a_i)^4 \phi_i(x_1), \quad x_1 \in (0, L) \\
&\phi_i(0) = \frac{\partial \phi_i}{\partial x_1}(0) = 0, \\
&\phi_i(L) = \frac{\partial \phi_i}{\partial x_1}(L) = 0, \\
&\int_0^L \phi_i^2(x_1)\, dx_1 = 1.
\end{aligned}$$

Let $\omega_i = \left(\frac{s_i}{L}\right)^2 \sqrt{\frac{Eh^3}{12(1-\nu^2)\rho^s h}}$ be the ith natural frequency.

The normal mode shapes ϕ_i for $i \in \mathbb{N}$ form an orthogonal basis of $L^2(0, L)$.

$$\eta(x_1, t) = \sum_{i \geq 0} \alpha_i(t)\phi_i(x_1), \quad u(x_1, t) = \sum_{i \geq 0} q_i(t)\phi_i(x_1)$$

where q_i is the solution of the second order differential equation

$$\begin{aligned} q_i''(t) + \omega_i^2 q_i(t) &= \frac{1}{\rho^s h}\alpha_i(t), \quad t \in (0, T) \\ q_i(0) &= \int_0^L u^0(x_1)\phi_i(x_1)\, dx_1 \\ q_i'(0) &= \int_0^L \dot{u}^0(x_1)\phi_i(x_1)\, dx_1. \end{aligned}$$

Newmark method

Knowing q_i^n, $\dot{q}_i^n$, $\ddot{q}_i^n$ and α_i^{n+1}, find q_i^{n+1}, $\dot{q}_i^{n+1}$, $\ddot{q}_i^{n+1}$ such that:

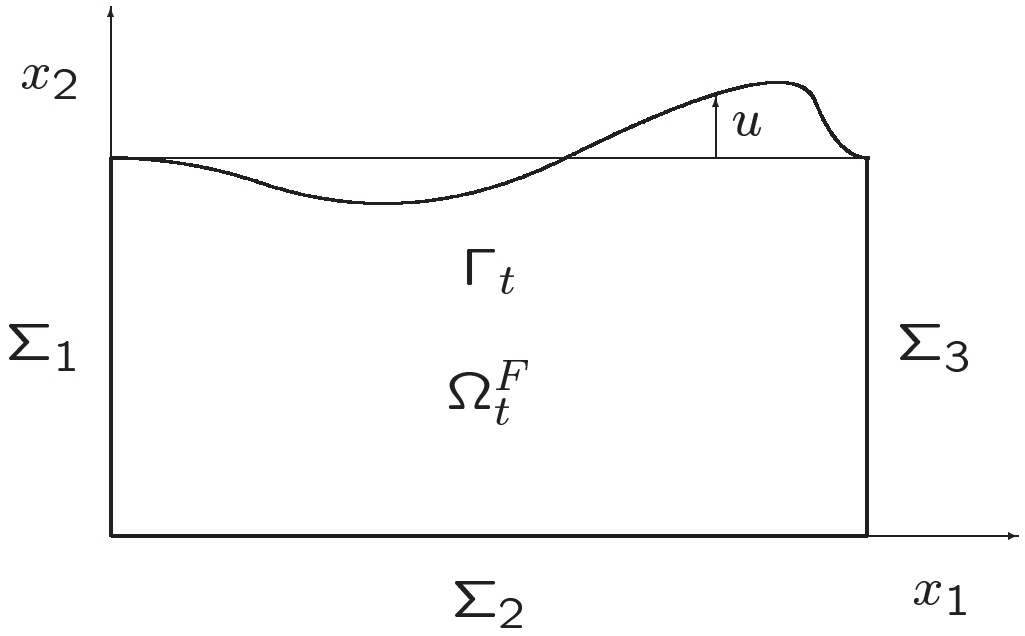
$$\ddot{q}_i^{n+1} + \omega_i^2 q_i^{n+1} = \frac{1}{\rho^s h}\alpha_i^{n+1},$$
$$\dot{q}_i^{n+1} = \dot{q}_i^n + \Delta t\left[(1-\delta)\ddot{q}_i^n + \delta\ddot{q}_i^{n+1}\right],$$
$$q_i^{n+1} = q_i^n + \Delta t\dot{q}_i^n + (\Delta t)^2\left[\left(\frac{1}{2}-\theta\right)\ddot{q}_i^n + \theta\ddot{q}_i^{n+1}\right]$$

where δ and θ are two real parameters.

$$\left(1+\omega_i^2(\Delta t)^2\theta\right)\ddot{q}_i^{n+1} = \frac{1}{\rho^s h}\alpha_i^{n+1} - \omega_i^2\left[q_i^n + \Delta t\dot{q}_i^n + (\Delta t)^2\left(\frac{1}{2}-\theta\right)\ddot{q}_i^n\right].$$

This method is unconditional stable for $2\theta \geq \delta \geq 1/2$. It is first order accuracy if $\delta \neq 1/2$. If $\delta = 1/2$, it is second order accuracy in the case $\theta \neq 1/12$ and forth order accuracy is achieved if $\theta = 1/12$.

Example of an admissible domain



Strong form of the unsteady Stokes equations

Find the velocity $\mathbf{v}(\mathbf{x},t) \in \mathbb{R}^2$ and the pressure $p(\mathbf{x},t) \in \mathbb{R}$ such that:

$$
\begin{aligned}
\rho^F \frac{\partial \mathbf{v}}{\partial t} - \mu \Delta \mathbf{v} + \nabla p &= \mathbf{f}^F, \quad \forall t \in (0,T), \forall \mathbf{x} \in \Omega_t^F \\
\nabla \cdot \mathbf{v} &= 0, \quad \forall t \in (0,T), \forall \mathbf{x} \in \Omega_t^F \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_1 \times (0,T) \\
p &= P_{in}, \quad \text{on } \Sigma_1 \times (0,T) \\
\mathbf{v} &= \mathbf{g}, \quad \text{on } \Sigma_2 \times (0,T) \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_3 \times (0,T) \\
p &= P_{out}, \quad \text{on } \Sigma_3 \times (0,T) \\
\mathbf{v}\left(x_1, H + u(x_1,t), t\right) &= \left(0, \frac{\partial u}{\partial t}(x_1,t)\right)^T, \quad \forall (x_1,t) \in (0,L) \times (0,T) \\
\mathbf{v}(\mathbf{x},0) &= \mathbf{v}^0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_0^F
\end{aligned}
$$

Arbitrary Lagrangian Eulerian (ALE) coordinates

$$\widehat{\Omega}^F = (0, L) \times (0, H) \qquad \widehat{\Gamma} = (0, L) \times \{H\}$$

$$\mathcal{A}_t\left(\widehat{x}_1, \widehat{x}_2\right) = \left(\widehat{x}_1, \frac{H + u\left(\widehat{x}_1, t\right)}{H} \widehat{x}_2\right)^T$$

$$\mathcal{A}_t^{-1}\left(x_1, x_2\right) = \left(x_1, \frac{H x_2}{H + u\left(x_1, t\right)}\right)^T$$

$$\boldsymbol{\vartheta}(\mathbf{x}) \stackrel{def}{=} \frac{\partial \mathcal{A}_t}{\partial t}(\widehat{\mathbf{x}})|_{t=t_{n+1}} = \left(0, \frac{\partial u}{\partial t}\left(x_1, t_{n+1}\right) \frac{x_2}{H + u\left(x_1, t_{n+1}\right)}\right)^T .$$

$$\frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t_{n+1}) \approx \frac{\mathbf{v}^{n+1}(\mathbf{x}) - \mathbf{v}^n\left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x})\right)}{\Delta t} - \left(\boldsymbol{\vartheta}(\mathbf{x}) \cdot \nabla\right) \mathbf{v}^{n+1}(\mathbf{x})$$

$$
\begin{aligned}
\rho^F \left(\frac{\mathbf{v}^{n+1}}{\Delta t} - (\boldsymbol{\vartheta} \cdot \nabla)\, \mathbf{v}^{n+1} \right) - \mu \Delta \mathbf{v}^{n+1} + \nabla p^{n+1} &= \rho^F \frac{\mathbf{v}^n \left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1} \right)}{\Delta t} + \mathbf{f}^F \\
&\quad \text{in } \Omega^F_{t_{n+1}} \\
\nabla \cdot \mathbf{v}^{n+1} &= 0 \text{ in } \Omega^F_{t_{n+1}} \\
\mathbf{v}^{n+1} \times \mathbf{n} &= 0 \text{ on } \Sigma_1 \\
p^{n+1} &= P_{in}(\cdot, t_{n+1}) \text{ on } \Sigma_1 \\
\mathbf{v}^{n+1} &= \mathbf{g}(\cdot, t_{n+1}) \text{ on } \Sigma_2 \\
\mathbf{v}^{n+1} \times \mathbf{n} &= 0 \text{ on } \Sigma_3 \\
p^{n+1} &= P_{out}(\cdot, t_{n+1}) \text{ on } \Sigma_3 \\
\mathbf{v}^{n+1} \left(x_1, H + u(x_1, t_{n+1}), t \right) &= \left(0, \frac{\partial u}{\partial t}(x_1, t_{n+1}) \right)^T, \\
&\quad 0 < x_1 < L.
\end{aligned}
$$

Mixed Finite Element

$$
\begin{aligned}
W^{n+1} &= \left\{ \mathbf{w} \in \left(H^1 \left(\Omega^F_{t_{n+1}} \right) \right)^2 ; \right. \\
&\quad \left. \mathbf{w} \times \mathbf{n} = 0 \text{ on } \Sigma_1 \cup \Sigma_3, \ \mathbf{w} = 0 \text{ on } \Sigma_2 \cup \Gamma_{t_{n+1}} \right\}, \\
Q^{n+1} &= L^2 \left(\Omega^F_{t_{n+1}} \right).
\end{aligned}
$$

Find the velocity $\mathbf{v}^{n+1}$ and the pressure p^{n+1} such that

$$
\left\{
\begin{array}{ll}
a_F^{n+1} \left(\mathbf{v}^{n+1}, \mathbf{w} \right) + d_F^{n+1} \left(\mathbf{v}^{n+1}, \mathbf{w} \right) + b_F^{n+1} \left(\mathbf{w}, p^{n+1} \right) = \ell^{n+1} \left(\mathbf{w} \right), \forall \mathbf{w} \\
b_F^{n+1} \left(\mathbf{v}^{n+1}, q \right) = 0, \forall q
\end{array}
\right.
$$

$$
\begin{aligned}
a_F^{n+1}\left(\mathbf{v}^{n+1},\mathbf{w}\right) &= \frac{\rho^F}{\Delta t}\left(\mathbf{v}^{n+1},\mathbf{w}\right) \\
&\quad +\mu\left(\nabla\times\mathbf{v}^{n+1},\nabla\times\mathbf{w}\right)+\mu\left(\nabla\cdot\mathbf{v}^{n+1},\nabla\cdot\mathbf{w}\right) \\
d_F^{n+1}\left(\mathbf{v}^{n+1},\mathbf{w}\right) &= -\rho^F\left(\left(\boldsymbol{\vartheta}\cdot\nabla\right)\mathbf{v}^{n+1},\mathbf{w}\right) \\
b_F^{n+1}\left(\mathbf{w},q\right) &= \left(\nabla\cdot\mathbf{w},q\right) \\
\ell^{n+1}\left(\mathbf{w}\right) &= \frac{\rho^F}{\Delta t}\left(\mathbf{v}^n\left(\mathcal{A}_{t_n}\circ\mathcal{A}_{t_{n+1}}^{-1}\right),\mathbf{w}\right)+\left(\mathbf{f}^F,\mathbf{w}\right) \\
&\quad -\int_{\Sigma_1}P_{in}(\cdot,t_{n+1})\mathbf{n}\cdot\mathbf{w}\,d\gamma-\int_{\Sigma_3}P_{out}(\cdot,t_{n+1})\mathbf{n}\cdot\mathbf{w}\,d\gamma
\end{aligned}
$$

Proposition 1 *If* $\|\nabla\cdot\mathbf{w}\|_{L^\infty}\leq\frac{1}{\Delta t}$*, then the bilinear form* $a_F^{n+1}+d_F^{n+1}$ *is elliptic.*

Strong form of the coupled equations

Find the transverse displacement u of the structure, the velocity $\mathbf{v}$ and the pressure p of the fluid such that

$$\eta(x_1,t) = -\left(\sigma^F \mathbf{n}\cdot\mathbf{e}_2\right)_{(x_1,H+u(x_1,t))}\sqrt{1+\left(\frac{\partial u}{\partial x_1}(x_1,t)\right)^2}$$

where $\sigma^F = -pI + \mu\left(\nabla\mathbf{v} + \nabla\mathbf{v}^T\right)$ is the stress tensor of the fluid, $\mathbf{e}_2 = (0,1)^T$ is the unit vector in the x_2 direction.

The displacement of the structure depends on the vertical component of the stresses exerced by the fluid on the interface. The movement of the structure changes the domain where the fluid equations must be solved. Also, on the interface we have to impose the equality between the fluid and structure velocity.

Identification of the stresses on the interface using the Least Squares Method

The stresses on the interface at the current time step t_{n+1} will be approached by $\eta_m^{n+1}(x_1) = \sum_{i=0}^{m-1} \alpha_i^{n+1} \phi_i(x_1)$.

The parameters α_i^{n+1} for $0 \leq i \leq m-1$ will be "identified" solving an optimization problem, more precisely

$$\boldsymbol{\alpha}^{n+1} \stackrel{def}{=} \left(\alpha_0^{n+1}, \ldots, \alpha_{m-1}^{n+1}\right) \in \arg \min_{\boldsymbol{\alpha} \in \mathbb{R}^m} J^{n+1}(\boldsymbol{\alpha}).$$

Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{R}^m$.

Structure sub-problem

For $i = 0, \ldots, m-1$, knowing q_i^n, $\dot{q}_i^n$, $\ddot{q}_i^n$, find Q_i, $\dot{Q}_i$, $\ddot{Q}_i$ such that:

$$\left(1 + \omega_i^2 (\Delta t)^2 \theta\right) \ddot{Q}_i = \frac{1}{\rho^s h} \alpha_i - \omega_i^2 \left[q_i^n + \Delta t \dot{q}_i^n + (\Delta t)^2 \left(\frac{1}{2} - \theta \right) \ddot{q}_i^n \right]$$

$$\dot{Q}_i = \dot{q}_i^n + \Delta t \left[(1-\delta) \ddot{q}_i^n + \delta \ddot{Q}_i \right],$$

$$Q_i = q_i^n + \Delta t \dot{q}_i^n + (\Delta t)^2 \left[\left(\frac{1}{2} - \theta \right) \ddot{q}_i^n + \theta \ddot{Q}_i \right]$$

Set

$$U(x_1) = \sum_{i=0}^{m-1} Q_i \phi_i(x_1), \; \dot{U}(x_1) = \sum_{i=0}^{m-1} \dot{Q}_i \phi_i(x_1), \; \ddot{U}(x_1) = \sum_{i=0}^{m-1} \ddot{Q}_i \phi_i(x_1).$$

Fluid sub-problem

$$
\begin{aligned}
W_h &= \Big\{ \mathbf{w}_h \in \left(C^0\left(\overline{\Omega}_h^F\right)\right)^2 ;\ \forall K \text{ triangle of } \mathcal{T}_h,\ \mathbf{w}_h|_K \in P1 + bubble, \\
&\quad \mathbf{w}_h \times \mathbf{n} = 0 \text{ on } \Sigma_1 \cup \Sigma_3,\ \mathbf{w}_h = 0 \text{ on } \Sigma_2 \cup \Gamma_h \Big\}, \\
Q_h &= \left\{ q_h \in C^0\left(\overline{\Omega}_h^F\right);\ \forall K \text{ triangle of } \mathcal{T}_h,\ q_h|_K \in P1 \right\}.
\end{aligned}
$$

Find the velocity $\mathbf{v}_h$ satisfies the boundary conditions

$$
\begin{aligned}
\mathbf{v}_h \times \mathbf{n} &= 0,\ on\ \Sigma_1 \cup \Sigma_3, \\
\mathbf{v}_h &\approx \mathbf{g}(\cdot, t_{n+1})\ on\ \Sigma_2, \\
\mathbf{v}_h\left(x_1, H + U(x_1)\right) &\approx \left(0, \dot{U}(x_1)\right)^T,\ 0 < x_1 < L
\end{aligned}
$$

and the pressure $p_h \in Q_h$ such that

$$
\begin{cases}
a_F^{n+1}\left(\mathbf{v}_h, \mathbf{w}_h\right) + d_F^{n+1}\left(\mathbf{v}_h, \mathbf{w}_h\right) + b_F^{n+1}\left(\mathbf{w}_h, p_h\right) = \ell^{n+1}\left(\mathbf{w}_h\right), & \forall \mathbf{w}_h \in W_h \\
b_F^{n+1}\left(\mathbf{v}_h, q_h\right) = 0, & \forall q_h \in Q_h
\end{cases}
$$

Definition of the cost function

Let us define for $i = 0, \ldots, m-1$

$$\beta_i = -\int_0^L \phi_i(x_1) \left(\sigma^F(\mathbf{v}_h, p_h)\,\mathbf{n}\cdot\mathbf{e}_2\right)_{(x_1, H+U(x_1))} \sqrt{1+\left(\frac{\partial U}{\partial x_1}(x_1)\right)^2}\, dx_1.$$

$$\beta_i = \int_0^L \phi_i(x_1) \left(p_h - \mu\left(\frac{\partial v_{h,1}}{\partial x_2} + \frac{\partial v_{h,2}}{\partial x_1}\right)\left(-\frac{\partial U}{\partial x_1}\right) - 2\mu\frac{\partial v_{h,2}}{\partial x_2}\right)_{(x_1, H+U(x_1))} dx_1.$$

Set the cost function

$$J^{n+1}(\boldsymbol{\alpha}) = \frac{1}{2}\sum_{i=0}^{m-1}(\alpha_i - \beta_i)^2.$$

$$\beta_i = \int_0^L \phi_i(x_1) p_h\left(x_1, H + U(x_1)\right) dx_1, \quad i = 0, \ldots, m-1.$$

Coupled fluid-structure algorithm by BFGS scheme

Step 1. Solve

$$\boldsymbol{\alpha}^{n+1} \in \arg \min_{\boldsymbol{\alpha} \in \mathbb{R}^m} J^{n+1}(\boldsymbol{\alpha})$$

using the Broyden, Fletcher, Goldforb, Shano (BFGS) scheme and set

$$\eta_m^{n+1}(x_1) = \sum_{i=0}^{m-1} \alpha_i^{n+1} \phi_i(x_1).$$

Step 2. Find q_i^{n+1}, $\dot{q}_i^{n+1}$, $\ddot{q}_i^{n+1}$ using the Newmark method and set

$$u_m^{n+1} = \sum_{i=0}^{m-1} q_i^{n+1} \phi_i, \quad \dot{u}_m^{n+1} = \sum_{i=0}^{m-1} \dot{q}_i^{n+1} \phi_i, \quad \ddot{u}_m^{n+1} = \sum_{i=0}^{m-1} \ddot{q}_i^{n+1} \phi_i.$$

Step 3. Build the mesh with triangular elements $\mathcal{T}_h^{n+1}$ by moving each node of $\widehat{\mathcal{T}}_h$ using the map

$$\mathcal{A}_{u_m^{n+1}}(\widehat{x}_1, \widehat{x}_2) = \left(\widehat{x}_1, \frac{H + u_m^{n+1}(\widehat{x}_1)}{H} \widehat{x}_2 \right)^T .$$

Step 4. Find $\mathbf{v}_h^{n+1}$ and p_h^{n+1} the solution of the generalized Stokes problem.

At each iteration of the BFGS scheme, the evaluation of the cost function and of its gradient in the current point is needed, also we have to solve an inaccurate line search.

BFGS scheme

Step 0 Choose a starting point $\alpha^0 \in \mathbb{R}^m$, an $m \times m$ symmetric positive matrix H_0 and a positive scalar ϵ. Set $k = 0$.

Step 1 Compute $\nabla J(\alpha^k)$.

Step 2 If $\left\|\nabla J(\alpha^k)\right\| < \epsilon$ stop.

Step 3 Set $\mathbf{d}^k = -H_k \nabla J(\alpha^k)$.

Step 4 Determine $\alpha^{k+1} = \alpha^k + \theta_k \mathbf{d}^k$, $\theta_k > 0$ by means of an approximate minimization

$$J(\alpha^{k+1}) \approx \min_{\theta \geq 0} J(\alpha^k + \theta \mathbf{d}^k).$$

Step 5 Compute $\delta_k = \alpha^{k+1} - \alpha^k$.

Step 6 Compute $\nabla J(\alpha^{k+1})$ and $\gamma_k = \nabla J(\alpha^{k+1}) - \nabla J(\alpha^k)$.

Step 7 Compute

$$H_{k+1} = H_k + \left(1 + \frac{\gamma_k^T H_k \gamma_k}{\delta_k^T \gamma_k}\right) \frac{\delta_k \delta_k^T}{\delta_k^T \gamma_k} - \frac{\delta_k \gamma_k^T H_k + H_k \gamma_k \delta_k^T}{\delta_k^T \gamma_k}$$

Step 8 Update $k = k + 1$ and go to the **Step 2**.

For the inaccurate line search at the **Step 4**, the methods of Goldstein and Armijo were used. If we denote by $g : [0, \infty) \to \mathbb{R}$ the function $g(\theta) = J(\alpha^k + \theta \mathbf{d}^k)$, we determine $\theta_k > 0$ such that

$$g(0) + (1 - \lambda)\, \theta_k g'(0) \leq g(\theta_k) \leq g(0) + \lambda \theta_k g'(0)$$

where $\lambda \in (0, 1/2)$.

We could compute $\nabla J(\alpha)$ by the Finite Differences Method

$$\frac{\partial J}{\partial \alpha_k}(\alpha) \approx \frac{J(\alpha + \Delta \alpha_k \mathbf{e_k}) - J(\alpha)}{\Delta \alpha_k}$$

where $\mathbf{e_k}$ is the k-th vector of the canonical base of $\mathbb{R}^m$ and $\Delta \alpha_k > 0$ is the grid spacing.

Numerical results

The computation has been made in a domain of length $L = 6\ cm$ and height $H = 1\ cm$.

The viscosity of the blood was taken to be $\mu = 0.035\ \frac{g}{cm \cdot s}$, its density $\rho^F = 1\ \frac{g}{cm^3}$. The thickness of the vessel is $h = 0.1\ cm$, the Young modulus $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the Poisson ratio $\nu = 0.5$, the density $\rho^S = 1.1\ \frac{g}{cm^3}$.

The volume force in fluid is $\mathbf{f}^F = (0, 0)^T$.

For the boundary conditions we have used:

$$
\begin{aligned}
P_{in}(\mathbf{x},t) &= \begin{cases} 10^3(1-\cos(\pi t/0.0025)), & \mathbf{x}\in\Sigma_1, 0\le t\le 0.005 \\ 0, & \mathbf{x}\in\Sigma_1, 0.005\le t\le T \end{cases} \\
\mathbf{g}(\mathbf{x},t) &= 0, \quad \mathbf{x}\in\Sigma_2, 0\le t\le T \\
P_{out}(\mathbf{x},t) &= 0, \quad \mathbf{x}\in\Sigma_3, 0\le t\le T
\end{aligned}
$$

and for the initial conditions we have taken:
$u^0 = 0$, $\dot{u}^0 = 0$, $\ddot{u}^0 = 0$, $\mathbf{v}^0 = 0$.

The numerical tests have been produced using *freefem++ v1.34*.

We have performed the simulation for $N = 737$ time steps with a time step $\Delta t = 0.0005\ s$ which gives a time duration $T = N\Delta t = 0.3685\ s$ longer that the one of the systolic phase of a cardiac beat (about 0.3 s).

Only the first 5 modes have been considered for the structure. In order to compute the normal mode shapes, the system has been solved using the software *Maple*. For the Newmark algorithm we have used $\delta = 0.6$ and $\theta = 0.5$.

For the approximation of the fluid velocity and pressure we have employed the triangular finite elements *P1 + bubble* and *P1* respectively.

The coupled fluid-structure algorithm is numerically stable for $\Delta t = 0.0005\ s$.

The behavior of this fluid-structure interaction problem is like a propagative system. The wave starts from the left side, the right side is reached after $t = 0.130\ s$ and comeback to the left side at $t = 0.280\ s$.

In Formaggia et al 2001, Nobile 2001, a 3D fluid-structure model is coupled with a 1D reduced model in order to reduce the reflexion due to the inappropriate boundary conditions for the structure and the fluid on the right side.

At each time step, the optimization problem have been solved by the BFGS algorithm, where the gradient of the cost function is approached by the Finite Differences Method with the grid spacing $\Delta\alpha_k = 0.1$.

We have performed 10 iterations of the BFGS algorithm when $0 < t_{n+1} \leq 0.005$ and 5 iteration when $t_{n+1} > 0.005$. The maximal number of the iterations was fixed to 5 in the method for the line search.

To sum up, at each time step the fluid sub-problem is solved at most

$$\text{no iterations BFGS} \times (1 + m + \text{no iterations line search}).$$

To reduce this number, an analytic formula for the gradient was employed in Murea 2003 and Murea, Vazquez 2004 in the steady case of a related fluid-structure interaction problem. In this case, the fluid sub-problem is solved at most

$$\text{no iterations BFGS} \times (1 + \text{no iterations line search}).$$

Also, we can inactivate the line search algorithm and we obtain that the fluid solver is called only 1 time at each BFGS iteration.

In Gerbeau, Vidrascu 2003 an efficient quasi-Newton algorithm is proposed where the tangent operator is approached.

Week form of the fluid equations in ALE coordinates

$$\begin{aligned}
\widehat{a}_F(\alpha,\widehat{\mathbf{v}},\widehat{\mathbf{w}}) &= \sum_{i=1}^{2}\int_{\Omega_0^F}\left(\frac{H+u(\widehat{x}_1)}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_1}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_1}-\frac{u'(\widehat{x}_1)\,\widehat{x}_2}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_1}\right)d\widehat{\mathbf{x}}\\
&+ \sum_{i=1}^{2}\int_{\Omega_0^F}\left(-\frac{u'(\widehat{x}_1)\,\widehat{x}_2}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_1}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}+\frac{H^2+\left(u'(\widehat{x}_1)\,\widehat{x}_2\right)^2}{H\left(H+u(\widehat{x}_1)\right)}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}\right)d\widehat{\mathbf{x}},\\
\widehat{b}_F(\alpha,\widehat{\mathbf{w}},\widehat{q}) &= -\int_{\Omega_0^F}\left(\frac{H+u(\widehat{x}_1)}{H}\frac{\partial\widehat{w}_1}{\partial\widehat{x}_1}-\frac{u'(\widehat{x}_1)\,\widehat{x}_2}{H}\frac{\partial\widehat{w}_1}{\partial\widehat{x}_2}+\frac{\partial\widehat{w}_2}{\partial\widehat{x}_2}\right)\widehat{q}\,d\widehat{\mathbf{x}}.
\end{aligned}$$

Find $\widehat{\mathbf{v}}\in\left(H^1\left(\Omega_0^F\right)\right)^2$, $\widehat{\mathbf{v}}|_{\Sigma}=\mathbf{g}$, $\widehat{\mathbf{v}}|_{\Gamma_0}=0$, $\widehat{p}\in\widehat{Q}$ *such that*

$$\begin{cases}
\widehat{a}_F(\alpha,\widehat{\mathbf{v}},\widehat{\mathbf{w}})+\widehat{b}_F(\alpha,\widehat{\mathbf{w}},\widehat{p}) &= \left\langle\widehat{\mathbf{f}}^F(\alpha),\widehat{\mathbf{w}}\right\rangle, & \forall\widehat{\mathbf{w}}\in\widehat{W}\\
\widehat{b}_F(\alpha,\widehat{\mathbf{v}},\widehat{q}) &= 0, & \forall\widehat{q}\in\widehat{Q}
\end{cases}$$

has a unique solution.

Analytic formula for the gradient in ALE coordinates

$$
\begin{aligned}
\frac{\partial \widehat{a}_F}{\partial \alpha_k}(\alpha, \widehat{\mathbf{v}}, \widehat{\mathbf{w}}) &= \sum_{i=1}^{2} \int_{\Omega_0^F} \left(\frac{u_k(\widehat{x}_1)}{H} \frac{\partial \widehat{v}_i}{\partial \widehat{x}_1} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_1} - \frac{u_k'(\widehat{x}_1)\,\widehat{x}_2}{H} \frac{\partial \widehat{v}_i}{\partial \widehat{x}_2} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_1} \right) d\widehat{\mathbf{x}} \\
&+ \sum_{i=1}^{2} \int_{\Omega_0^F} \left(-\frac{u_k'(\widehat{x}_1)\,\widehat{x}_2}{H} \frac{\partial \widehat{v}_i}{\partial \widehat{x}_1} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_2} \right) d\widehat{\mathbf{x}} \\
&+ \sum_{i=1}^{2} \int_{\Omega_0^F} \left(\frac{2u_k'(\widehat{x}_1)\,u'(\widehat{x}_1)\,(\widehat{x}_2)^2}{H\,(H + u(\widehat{x}_1))} \frac{\partial \widehat{v}_i}{\partial \widehat{x}_2} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_2} \right) d\widehat{\mathbf{x}} \\
&+ \sum_{i=1}^{2} \int_{\Omega_0^F} \left(\frac{-u_k(\widehat{x}_1)\left(H^2 + (u'(\widehat{x}_1)\,\widehat{x}_2)^2\right)}{H\,(H + u(\widehat{x}_1))^2} \frac{\partial \widehat{v}_i}{\partial \widehat{x}_2} \frac{\partial \widehat{w}_i}{\partial \widehat{x}_2} \right) d\widehat{\mathbf{x}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \widehat{b}_F}{\partial \alpha_k}\left(\alpha, \widehat{\mathbf{w}}, \widehat{q}\right) &= -\int_{\Omega_0^F}\left(\frac{u_k\left(\widehat{x}_1\right)}{H}\frac{\partial \widehat{w}_1}{\partial \widehat{x}_1} - \frac{u_k'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial \widehat{w}_1}{\partial \widehat{x}_2}\right)\widehat{q}\, d\widehat{\mathbf{x}} \\
\frac{\partial}{\partial \alpha_k}\left\langle \widehat{\mathbf{f}}^F\left(\alpha\right), \widehat{\mathbf{w}}\right\rangle &= \sum_{i=1}^{2}\int_{\Omega_0^F}\frac{u_k\left(\widehat{x}_1\right)}{H} f_i^F \widehat{w}_i\, d\widehat{\mathbf{x}}.
\end{aligned}
$$

$$
\frac{\partial J}{\partial \alpha_k}(\alpha) = \int_0^L\left(\phi_k - \int_0^L \frac{\phi_k}{L} dx_1 - \frac{\partial \widehat{p}_0}{\partial \alpha_k}\right)\left(\sum_{i=1}^{m}\alpha_i\left(\phi_i - \int_0^L \frac{\phi_i}{L} dx_1\right) - \widehat{p}_0\right) dx_1
$$