The BFGS algorithm for a nonlinear least squares problem arising from blood flow in arteries

C. M. Murea
Laboratoire de Mathématiques et Applications, Université de Haute-Alsace,
4, rue des Frères Lumière, 68093 MULHOUSE Cedex, France,
e-mail: c.murea@uha.fr
http://www.math.uha.fr/~edp/murea/

Abstract

Using the Arbitrary Lagrangian Eulerian coordinates and the Least Squares Method, a two dimensional steady fluid structure interaction problem is transformed in an optimal control problem. Sensitivity analysis is presented. The BFGS algorithm gives satisfactory numerical results even when we use a reduced number of discrete controls.

keywords: fluid structure interaction, Arbitrary Lagrangian Eulerian coordinates, BFGS algorithm

1 Introduction

In this paper we consider a two dimensional fluid structure interaction. The mathematical model which governs the fluid is the steady Stokes equations, while the structure verifies the beam equation which does not involve shearing stress. The solution of the model is given by the displacement of the structure, the velocity and the pressure of the fluid. The boundary of the fluid admits the following decomposition: a moving part, which represents the interface between the fluid and the structure, and a rigid part. This kind of problem is of considerable interest in the simulation of blood flow in large arteries (see [1], [2], [3]) or in aeroelasticity (see [4]).

The existence results for the fluid structure interaction can be found in [5], [6] for the steady case and in [7], [8], [9] for the unsteady case.

Sensitivity analysis of a coupled fluid structure system was investigated in [10].

The asymptotic limit when the fluid domain width approaches to zero can be modeled by a one dimensional model of Stokes equation, widely used in lubrication theory (see [11]).

In a previous work ([12]), a three dimensional fluid structure interaction was formulated as an optimal control system, where the control is the force acting on the interface and the observation is the velocity of the fluid on the interface. The fluid equations were solved taking into account a given surface force on the interface.

A similar approach was used in [13], where it was proved that the cost function is differentiable. The analytic computation of the gradient for the cost function is important because it enables us to apply accurate numerical methods (see [14]). The exact gradient of the cost function is computed in [13].

Numerical results for a two dimensional fluid structure interaction using the optimal control method are presented in [15]. The fluid equations are solved subject to the conditions of zero normal velocity and a given value of pressure on the interface. The control is the value of the pressure at the interface and the observation is the tangential velocity on the interface.

The most frequently, the fluid-structure interaction problems are solved numerically by partitioned procedures, i.e. the fluid and the structure equations are solved separately, which enables us to use the existing solvers for each sub-problem.

This can be done using fixed point strategies with eventually a relaxation parameter, but these methods do not always converge or they have slow convergence rate [16], [17], [1]. The convergence can be accelerated using Aitken's method [2] or transpiration condition [18].

Other way to accelerate the convergence is to use methods which employ the derivative. In [19] a block Newton algorithm was used where the derivative of the operators are approached by finite differences. Good convergence rate was obtained in [2] where the derivative of the operator was replaced by a simpler operator. At each time step, a quasi-Newton algorithm was used to solve a fluid-structure interaction problem. The mean number of iterations of the quasi-Newton algorithm is 6.1. With the Aitken acceleration method this number is 24.1. At each iteration, a Stokes and a Laplacian problems were solved in the current fluid domain.

In the present work, a fluid structure interaction problem was formulated as an optimal control system, where the control is the force acting on the interface and the observation is the pressure on the interface. The boundary condition to be imposed on the fluid is that all components of the velocity are zero at the interface.

To solve numerically the optimal control problem, we use a quasi Newton method which employs the analytic gradient of the cost function and the approximation of the inverse Hessian is updated by the Broyden, Fletcher, Goldforb, Shano (BFGS) scheme. This algorithm is faster than fixed point with relaxation or block Newton methods which represents the main advantage of using the optimal control approach for fluid-structure interaction problem. The finite element functions of the normal stresses at the interface are not necessary the same as the trace on the interface of the pressure finite element functions. This is another advantage by comparison with the fixed point approach.

An outline of the paper is as follows. First, we prove that the normal force acting on the structure depends only on the pressure. Then, an exact solution for a particular fluid structure interaction is given. Using the Least Square Method, the fluid structure interaction will be reformulated as an optimal control problem. We will analyse the dependence of the displacement of the interface, the velocity, the pressure of the fluid and the cost function on variations of the discrete control. Finally, numerical results are presented.

2 Notations

Let L and H be two positive constants. We define the set

$$\begin{aligned}\mathcal{U}_{ad} = & \Big\{u \in \mathcal{C}^1\left([0,L]\right); u\left(0\right) = u\left(L\right) = u'\left(0\right) = u'\left(L\right) = 0, \\ & \textstyle\int_0^L u(x_1)\,dx_1 = 0,\ \inf_{x_1 \in [0,L]} \left\{H + u\left(x_1\right)\right\} > 0\Big\}\end{aligned}$$

where u' is the first derivative of u.

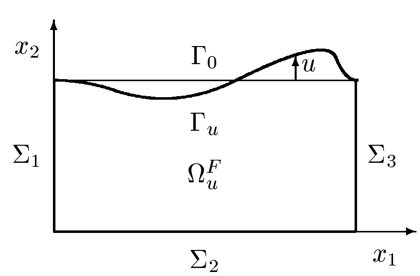

Figure 1: Sets appearing in the fluid-structure problem

For each $u \in \mathcal{U}_{ad}$, we introduce the notations (see Figure 1)

$$\begin{aligned}
\Omega_u^F &= \left\{(x_1, x_2) \in \mathbb{R}^2;\, x_1 \in (0, L),\, 0 < x_2 < H + u(x_1)\right\},\\
\Gamma_u &= \left\{(x_1, x_2) \in \mathbb{R}^2;\, x_1 \in (0, L),\, x_2 = H + u(x_1)\right\}.
\end{aligned}$$

Also, we denote

$$\begin{aligned}
\Sigma_1 &= \left\{(0, x_2) \in \mathbb{R}^2;\, x_2 \in (0, H)\right\}\\
\Sigma_2 &= \left\{(x_1, 0) \in \mathbb{R}^2;\, x_1 \in (0, L)\right\}\\
\Sigma_3 &= \left\{(L, x_2) \in \mathbb{R}^2;\, x_2 \in (0, H)\right\}.
\end{aligned}$$

The two-dimensional domain occupied by the fluid is Ω_u^F, the interface between the fluid and the structure is Γ_u, while $\Sigma = \Sigma_1 \cup \overline{\Sigma_2} \cup \Sigma_3$ represents the rigid boundary of the fluid.

In the following, we denote by $\mathbf{n} = (n_1, n_2)^T$ the unit outward normal vector and by $\tau = (\tau_1, \tau_2)^T = (-n_2, n_1)^T$ the unit tangential vector to $\partial\Omega_u^F$.

3 Position of the problem

We suppose that the fluid is governed by the steady Stokes equations, while the deformation of the elastic part of the boundary verifies a particular beam equation which does not involve shearing stress (see [20]). We consider that the structure is a beam of axis parallel to Ox_1 with constant thickness h. We assume that the displacement of the beam is normal to its axis.

The problem is to find:

- $u : [0, L] \to \mathbb{R}$ the displacement of the structure,
- $\mathbf{v} = (v_1, v_2)^T : \Omega_u^F \to \mathbb{R}^2$ the velocity of the fluid and
- $p : \Omega_u^F \to \mathbb{R}$ the pressure of the fluid,

such that

$$\begin{aligned}
u''''(x_1) &= \frac{1}{D}\left(f^S(x_1) + p\left(x_1, H + u(x_1)\right)\right), \quad \forall x_1 \in (0, L) && (1)\\
u(0) &= u(L) = u'(0) = u'(L) = 0 && (2)\\
\int_0^L u(x_1)\, dx_1 &= 0 && (3)\\
0 &< \inf_{x_1 \in [0, L]} \left\{H + u(x_1)\right\} && (4)\\
-\mu\Delta\mathbf{v} + \nabla p &= \mathbf{f}^F, \quad in\ \Omega_u^F && (5)\\
\operatorname{div} \mathbf{v} &= 0, \quad in\ \Omega_u^F && (6)\\
\mathbf{v} &= \mathbf{g}, \quad on\ \Sigma && (7)\\
\mathbf{v} &= 0, \quad on\ \Gamma_u && (8)
\end{aligned}$$

where

- $D = \frac{Eh^3}{12}$ is a structure constant, E is the Young modulus, h is the thickness.
- $f^S : (0, L) \to \mathbb{R}$ are the averaged volume forces of the structure, in general the gravity forces and in this case we have $f^S(x_1) = -g_0 \rho^S h$, where g_0 is the gravity, ρ^S is the density of the structure,
- $\mu > 0$ is the viscosity of the fluid,

- $\mathbf{f}^F = (f_1^F, f_2^F)^T : \Omega_u^F \to \mathbb{R}^2$ are the volume forces of the fluid, in general the gravity forces,
- $\mathbf{g} = (g_1, g_2)^T : \Sigma \to \mathbb{R}^2$ is the imposed velocity profile of the fluid on the rigid boundary, such that

$$\int_\Sigma \mathbf{g} \cdot \mathbf{n} \, d\sigma = 0. \tag{9}$$

The incompressibility of the fluid (6) together with the boundary conditions (7), (8) and the relation (9) imply that the volume of the fluid is conserved or equivalently $\int_0^L u(x_1)\, dx_1$ is constant. Without loss of generality, we assume that this constant is zero and we obtain the condition (3).

The inequality (4) states that the fluid domain is connected.

For the Newtonian fluids, the stress tensor σ has the form

$$\sigma = -p\, I + \mu \left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right),$$

consequently, the fluid forces acting on the structure are $-\sigma \mathbf{n}$.

Proposition 1 *If* $\mathbf{v} \in \left(H^2 \left(\Omega_u^F \right) \right)^2$, $p \in H^1 \left(\Omega_u^F \right)$, $\mathbf{v}$ *is constant on* Γ_u, $\operatorname{div} \mathbf{v} = 0$ *in* Ω_u^F, *then* $-(\sigma \mathbf{n}) \cdot \mathbf{n} = p$ *on* Γ_u.

Proof. This result is a corollary of the Proposition 3.1 from [21] and it is similar to the Proposition 4.5 from the same paper. We have that

$$-(\sigma \mathbf{n}) \cdot \mathbf{n} = p - \mu \left(\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) \mathbf{n} \right) \cdot \mathbf{n}$$

and

$$\nabla \mathbf{v} + \nabla \mathbf{v}^T = \begin{pmatrix} 2\frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} & 2\frac{\partial v_2}{\partial x_2} \end{pmatrix}.$$

It is follows that

$$\left(\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) \mathbf{n} \right) \cdot \mathbf{n} = 2\frac{\partial v_1}{\partial x_1} n_1^2 + 2 \left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) n_1 n_2 + 2 \frac{\partial v_2}{\partial x_2} n_2^2.$$

In Proposition 3.1 from [21], it is proved that $\frac{\partial v_i}{\partial x_j} n_k = \frac{\partial v_i}{\partial x_k} n_j$, $\forall i, j, k \in \{1, 2\}$, so $\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1} \right) n_1 n_2 = \frac{\partial v_1}{\partial x_1} n_2^2 + \frac{\partial v_2}{\partial x_2} n_1^2$ and this implies that

$$\left(\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T \right) \mathbf{n} \right) \cdot \mathbf{n} = 2\frac{\partial v_1}{\partial x_1} \left(n_1^2 + n_2^2 \right) + 2\frac{\partial v_2}{\partial x_2} \left(n_1^2 + n_2^2 \right) = 2 \operatorname{div} \mathbf{v} = 0$$

which ends the proof. $\square$

Under the assumption of small displacement of the beam, it follows that $\mathbf{n} \approx (0, 1)^T$. Then, it is reasonable to solve the beam equation (1) under the fluid forces given by $p\,(x_1, H + u(x_1))$, $x_1 \in (0, L)$.

4 Exact solution for a particular case

We assume that the density of the fluid is constant ρ^F and the volume forces in the fluid have the form $\mathbf{f}^F = (0, -\rho^F g_0)^T$, where g_0 is the gravitational acceleration. The velocity profile of the fluid on the rigid boundary is given by:

$$\begin{aligned} g_1(x_1, x_2) &= \begin{cases} \left(1 - \frac{x_2^2}{H^2} \right) V_0, & (x_1, x_2) \in \Sigma_1 \cup \Sigma_3 \\ V_0, & (x_1, x_2) \in \Sigma_2 \end{cases} \\ g_2(x_1, x_2) &= 0, \quad (x_1, x_2) \in \Sigma. \end{aligned}$$

We assume that the density of the structure ρ^S and its thickness h are constant.
We assume that the averaged volume forces in the structure have the form

$$f^S(x_1) = -\rho^S g_0\, h + \frac{2\mu V_0}{H^2} x_1, \quad \forall x_1 \in (0, L). \tag{10}$$

Then, we have the following solution for the system (1)–(8):

$$\begin{aligned}
u(x_1) &= 0, \quad \forall x_1 \in (0, L) \\
v_1(x_1, x_2) &= \left(1 - \tfrac{x_2^2}{H^2}\right) V_0, \quad \forall (x_1, x_2) \in \Omega_u^F \\
v_2(x_1, x_2) &= 0, \quad \forall (x_1, x_2) \in \Omega_u^F \\
p(x_1, x_2) &= \rho^S g_0\, h - \tfrac{2\mu V_0}{H^2} x_1 + \rho^F g_0\, (H - x_2)\,, \quad \forall (x_1, x_2) \in \Omega_u^F.
\end{aligned}$$

Remark 1 *The term $\frac{2\mu V_0}{H^2} x_1$ in (10) is artificial. It was added to obtain a solution where the displacement of the beam is null and the flow is Poiseuille.*

5 Fixed point approach

We start with a result concerning the equations of the interface.

Proposition 2 *For a given continuous function $\eta : [0, L] \to \mathbb{R}$ there exist a unique function $u : [0, L] \to \mathbb{R}$ of class $\mathcal{C}^4$ and a unique constant $c \in \mathbb{R}$ solutions of*

$$u''''(x_1) = \frac{1}{D} \left(\eta(x_1) + c\right), \quad \forall x_1 \in (0, L) \tag{11}$$

with boundary conditions (2), such that the equality (3) holds.

Proof. *Existence.* Let $u_\eta : [0, L] \to \mathbb{R}$ be the unique solution of

$$u''''(x_1) = \frac{1}{D} \eta(x_1), \quad \forall x_1 \in (0, L)$$

with boundary conditions (2). The unique solution of

$$u''''(x_1) = \frac{1}{D}, \quad \forall x_1 \in (0, L)$$

with boundary conditions (2) is $x_1 \in [0, L] \mapsto \frac{x_1^2 (x_1 - L)^2}{24D} \in \mathbb{R}$.

Then, the solutions of (11) and (2) have the form

$$u(x_1) = u_\eta(x_1) + c \frac{x_1^2 (x_1 - L)^2}{24D}.$$

The equality (3) is equivalent to

$$\int_0^L u_\eta(x_1)\, dx_1 + c \int_0^L \frac{x_1^2 (x_1 - L)^2}{24D}\, dx_1 = 0$$

consequently, if we set

$$c = -\frac{720D}{L^5} \int_0^L u_\eta(x_1)\, dx_1,$$

then the condition (3) holds.

Uniqueness. Let u_i, c_i, $i = 1, 2$ be two solutions of (11), such that $\int_0^L u_i\, dx_1 = 0$. By subtracting, we obtain that

$$(u_1 - u_2)''''(x_1) = \frac{1}{D} (c_1 - c_2)\,, \quad \forall x_1 \in (0, L)$$

and $u_1 - u_2$ verifies the boundary conditions (2). Consequently we have $(u_1 - u_2)(x_1) = (c_1 - c_2)\,\frac{x_1^2(x_1-L)^2}{24D}$. Since $\int_0^L (u_1 - u_2) dx_1 = 0$, we obtain $c_1 - c_2 = 0$ and $u_1 - u_2 = 0$. $\square$

From the above Proposition, it follows that for a given continuous function $\lambda_0 : (0, L) \to \mathbb{R}$, such that $\int_0^L \lambda_0(x_1)\, dx_1 = 0$, we can solve the beam equations

$$u''''(x_1) = \frac{1}{D}\left(f^S(x_1) + \lambda_0(x_1) + c\right), \quad \forall x_1 \in (0, L) \tag{12}$$

with boundary conditions (2) where c is the real constant such that the equality (3) holds.

Let $\mathcal{S}$ be defined by

$$\mathcal{S}(\lambda_0) = u. \tag{13}$$

If $0 < \inf_{x_1 \in [0,L]} \{H + u(x_1)\}$, we can solve the Stokes equations (5)–(8) and we obtain v and p. The pressure is determined up to an additive constant, i.e. it has the form $p = p_0 + C$, where p_0 is a particular solution and C is a real constant. We will take p_0 such that $\int_0^L p_0(x_1, H + u(x_1))\, dx_1 = 0$.

We denote by $\mathcal{F}(u)$ the function

$$x_1 \in (0, L) \mapsto p_0(x_1, H + u(x_1)). \tag{14}$$

The function $\mathcal{F}(u)$ is well defined, if the trace of the pressure p_0 on Γ_u exits. For this, we have to precise the regularity of the solution of Stokes equations.

Let $\overline{\mathbf{g}} : \partial\Omega_u^F \to \mathbb{R}$ be defined by $\overline{\mathbf{g}}(\mathbf{x}) = 0$ for $\mathbf{x} \in \Gamma_u$ and $\overline{\mathbf{g}}(\mathbf{x}) = \mathbf{g}(\mathbf{x})$ for $\mathbf{x} \in \Sigma$.

If $\partial\Omega_u^F$ is Lipschitz continuous, Ω_u^F is a connected domain, $\mathbf{f}^F \in \left(H^{-1}\left(\Omega_u^F\right)\right)^2$, $\overline{\mathbf{g}} \in \left(H^{1/2}\left(\partial\Omega_u^F\right)\right)^2$ such that $\int_{\partial\Omega_u^F} \overline{\mathbf{g}} \cdot \mathbf{n}\, d\sigma = 0$, then the problem :
find $\mathbf{v} \in \left(H^1\left(\Omega_u^F\right)\right)^2$, $\mathbf{v} = \overline{\mathbf{g}}$ on $\partial\Omega_u^F$ and $p \in L^2\left(\Omega_u^F\right)/\mathbb{R}$

$$\left\{\begin{array}{lcll} \displaystyle\int_{\Omega_u^F} \nabla\mathbf{v} \cdot \nabla\mathbf{w}\, d\mathbf{x} - \int_{\Omega_u^F} (\mathrm{div}\ \mathbf{w})\, p\, d\mathbf{x} & = & \displaystyle\int_{\Omega_u^F} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x}, & \forall \mathbf{w} \in \left(H_0^1\left(\Omega_u^F\right)\right)^2 \\ \displaystyle -\int_{\Omega_u^F} (\mathrm{div}\ \mathbf{v})\, q\, d\mathbf{x} & = & 0, & \forall q \in L^2\left(\Omega_u^F\right)/\mathbb{R} \end{array}\right. \tag{15}$$

has a unique solution.

Moreover, if $\partial\Omega_u^F$ is of class $\mathcal{C}^2$, $\mathbf{f}^F \in \left(L^2\left(\Omega_u^F\right)\right)^2$ and $\overline{\mathbf{g}} \in \left(H^{3/2}\left(\partial\Omega_u^F\right)\right)^2$, then $\mathbf{v} \in \left(H^2\left(\Omega_u^F\right)\right)^2$ and $p \in H^1\left(\Omega_u^F\right)/\mathbb{R}$.

These results could be found in [22, p. 88].

The fixed point approach is to find λ_0 such that $\mathcal{F} \circ \mathcal{S}(\lambda_0) = \lambda_0$, where $\mathcal{S}$ and $\mathcal{F}$ are defined by (13) and (14).

The existence of a fixed point will not be treated here. It is important to note that if we want to apply the Schauder's fixed point theorem, the regularity of λ_0 and $\mathcal{F} \circ \mathcal{S}(\lambda_0)$ must be the same. It is not the case in our framework: for $\lambda_0 \in \mathcal{C}^0(0, L)$, we have $\mathcal{S}(\lambda_0) = u \in \mathcal{C}^4(0, L)$ and consequently $\mathcal{F}(u) \in H^{1/2}(0, L)$. It is known that $H^{1/2}(0, L)$ is not included in $\mathcal{C}^0(0, L)$, but $H^{1/2+\epsilon}(0, L) \subset \mathcal{C}^0(0, L)$ for $\epsilon > 0$. Existence results for related steady fluid-structure interaction problems can be found in [5] and [6].

In the following, we relax the fixed point problem by the Least Squares Method and we obtain an optimization problem.

6 Least Squares approach

Let $\phi_i : [0, L] \to \mathbb{R}$ be some particular given functions and α_i are the scalar parameters to be identified, $1 \leq i \leq m$.

Let us comment the regularity and the shape of ϕ_i. We take $\phi_i \in \mathcal{C}^0(0, L)$, the condition $\int_0^L \phi_i(x_1)dx_1 = 0$ is not necessary needed. Also, the functions ϕ_i are not necessary the same that the trace on the interface of the pressure finite element functions. This is an advantage by comparison with the fixed point approach.

For given $\alpha = (\alpha_1, \ldots, \alpha_m)$, we find $u : [0, L] \to \mathbb{R}$ and $c(\alpha) \in \mathbb{R}$ solutions of

$$u''''(x_1) = \frac{1}{D}\left(f^S(x_1) + \sum_{i=1}^{m} \alpha_i \phi_i(x_1) + c(\alpha)\right), \quad \forall x_1 \in (0, L) \tag{16}$$

with boundary conditions (2), such that (3) holds.

The next step is to solve the Stokes equations in the domain Ω_u^F and we obtain $\mathbf{v}$ and p. We assume that $p \in H^1\left(\Omega_u^F\right)$ and we set $p_0 = p - \frac{1}{L}\int_0^L p\left(x_1, H + u(x_1)\right)dx_1$. It follows that

$$\int_0^L p_0\left(x_1, H + u(x_1)\right)dx_1 = 0. \tag{17}$$

Let $J : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$J(\alpha) = \int_0^L \left(\sum_{i=1}^{m} \alpha_i \left(\phi_i(x_1) - \frac{1}{L}\int_0^L \phi_i(x_1)dx_1\right) - p_0\left(x_1, H + u(x_1)\right)\right)^2 dx_1.$$

Now, the problem is to find $\alpha \in \mathbb{R}^m$ solution of

$$\begin{cases} \inf J(\alpha) \\ u \textit{ solution of } (16),\ (2),\ (3), \\ u \textit{ verifies } (4), \\ \mathbf{v},\ p_0 \textit{ solution of } (5)-(8), \\ p_0 \textit{ verifies } (17). \end{cases} \tag{18}$$

In other words, we try to find a solution of the system (1)–(8) such that

$$p\left(x_1, H + u(x_1)\right) \approx \sum_{i=1}^{m} \alpha_i \phi_i(x_1) + c(\alpha), \quad \forall x_1 \in (0, L)$$

where $\alpha \in \mathbb{R}^m$ and $p(x_1, x_2) = p_0(x_1, x_2) + \sum_{i=1}^{m} \frac{\alpha_i}{L}\int_0^L \phi_i(x_1)dx_1 + c(\alpha)$, for $(x_1, x_2) \in \Omega_u^F$.

The discrete control is $\alpha \in \mathbb{R}^m$ and the observation is the trace of the pressure on the interface, more precisely

$$x_1 \in (0, L) \to p_0\left(x_1, H + u(x_1)\right).$$

7 Sensitivity analysis

We shall analyse the dependence of the displacement of the interface u, the velocity, the pressure of the fluid $\mathbf{v}$, p and the cost function J on variations of the discrete control α.

7.1 Sensitivity of the displacement of the interface

Proposition 3 *The applications $\alpha \to u$ and $\alpha \to c(\alpha)$ are affine, where u and $c(\alpha)$ are the solutions of the equation (16) with boundary conditions (2), such that (3) holds. More precisely,*

$$\begin{array}{rcl} u & = & u_0 + \sum_{i=1}^{m} \alpha_i u_i \\ c(\alpha) & = & c_0 + \sum_{i=1}^{m} \alpha_i c_i \end{array}$$

where u_0, c_0 *verify*

$$\left\{\begin{array}{rcl} u_0''''(x_1) & = & \frac{1}{D}\left(f^S(x_1)+c_0\right), \quad \forall x_1 \in (0,L) \\ u_0\,(0) & = & u_0\,(L) = u_0'\,(0) = u_0'\,(L) = 0 \\ \int_0^L u_0(x_1)\,dx_1 & = & 0 \end{array}\right. \tag{19}$$

and u_i, c_i *verify*

$$\left\{\begin{array}{rcl} u_i''''(x_1) & = & \frac{1}{D}\left(\phi_i(x_1)+c_i\right), \quad \forall x_1 \in (0,L) \\ u_i\,(0) & = & u_i\,(L) = u_i'\,(0) = u_i'\,(L) = 0 \\ \int_0^L u_i(x_1)\,dx_1 & = & 0. \end{array}\right. \tag{20}$$

Proof. According to Proposition 2, the systems (19) and (20) have unique solutions. By addition, we obtain

$$\left(u_0+\sum_{i=1}^m \alpha_i u_i\right)''''(x_1) = \frac{1}{D}\left(f^S(x_1)+\sum_{i=1}^m \alpha_i\phi_i(x_1)+c_0+\sum_{i=1}^m \alpha_i c_i\right).$$

Also, the application $x_1 \mapsto \left(u_0+\sum_{i=1}^m \alpha_i u_i\right)(x_1)$ verifies the boundary conditions (2) and $\int_0^L \left(u_0+\sum_{i=1}^m \alpha_i u_i\right)(x_1)\,dx_1 = 0$. From the Proposition 2 and the definition of u and $c(\alpha)$ by (16), (2), (3), we obtain the conclusion. $\square$

7.2 Sensitivity of the velocity and the pressure of the fluid

In order to study the sensitivity of the velocity and the pressure of the fluid we follow [13] where the Arbitrary Lagrangian Eulerian (ALE) coordinates have been used.

We denote by $\Omega_0^F = (0,L)\times(0,H)$ the reference domain and by $\Gamma_0 = (0,L)\times\{H\}$ its top boundary. For each $u \in \mathcal{U}_{ad}$ we consider the following one-to-one continuous differentiable transformation $T_u : \overline{\Omega_0^F} \to \overline{\Omega_u^F}$ given by:

$$T_u\,(\widehat{x}_1,\widehat{x}_2) = \left(\widehat{x}_1, \frac{H+u\,(\widehat{x}_1)}{H}\widehat{x}_2\right)$$

which admits the continuous differentiable inverse mapping

$$T_u^{-1}\,(x_1,x_2) = \left(x_1, \frac{Hx_2}{H+u\,(x_1)}\right)$$

and verifies that $T_u\left(\Omega_0^F\right) = \Omega_u^F$, $T_u\,(\Gamma_0) = \Gamma_u$ and $T_u\,(\widehat{x}) = \widehat{x}$, $\forall \widehat{x} \in \Sigma$.

We set $\mathbf{x} = T_u\,(\widehat{\mathbf{x}})$ for each $\mathbf{x} = (x_1,x_2) \in \Omega_u^F$ and $\widehat{\mathbf{x}} = (\widehat{x}_1,\widehat{x}_2) \in \Omega_0^F$.

We denote by $\widehat{\mathbf{v}}(\widehat{\mathbf{x}}) = \mathbf{v}\,(T_u(\widehat{\mathbf{x}}))$ and $\widehat{p}(\widehat{\mathbf{x}}) = p\,(T_u(\widehat{\mathbf{x}}))$ the velocity and the pressure in the reference domain Ω_0^F.

In order to pose the variational formulation in the reference configuration let us consider the following Hilbert spaces:

$$\begin{array}{rcl} \widehat{W} & = & \left(H_0^1\left(\Omega_0^F\right)\right)^2 \\ \widehat{Q} & = & L^2\left(\Omega_0^F\right)/\mathbb{R} \end{array}$$

equipped with their usual inner products. We introduce the forms

$$\widehat{a}_F : \mathbb{R}^m \times \left(H^1\left(\Omega_0^F\right)\right)^2 \times \left(H^1\left(\Omega_0^F\right)\right)^2 \to \mathbb{R} \qquad \widehat{b}_F : \mathbb{R}^m \times \left(H^1\left(\Omega_0^F\right)\right)^2 \times \widehat{Q} \to \mathbb{R}$$

defined by

$$\begin{aligned}
\widehat{a}_F\left(\alpha,\widehat{\mathbf{v}},\widehat{\mathbf{w}}\right) &= \sum_{i=1}^{2}\int_{\Omega_0^F}\left(\frac{H+u\left(\widehat{x}_1\right)}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_1}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_1}-\frac{u'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_1}\right)d\widehat{\mathbf{x}}\\
&+\sum_{i=1}^{2}\int_{\Omega_0^F}\left(-\frac{u'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_1}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}+\frac{H^2+\left(u'\left(\widehat{x}_1\right)\widehat{x}_2\right)^2}{H\left(H+u\left(\widehat{x}_1\right)\right)}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}\right)d\widehat{\mathbf{x}},\\
\widehat{b}_F\left(\alpha,\widehat{\mathbf{w}},\widehat{q}\right) &= -\int_{\Omega_0^F}\left(\frac{H+u\left(\widehat{x}_1\right)}{H}\frac{\partial\widehat{w}_1}{\partial\widehat{x}_1}-\frac{u'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial\widehat{w}_1}{\partial\widehat{x}_2}+\frac{\partial\widehat{w}_2}{\partial\widehat{x}_2}\right)\widehat{q}\,d\widehat{\mathbf{x}}.
\end{aligned}$$

We assume that the volume forces in fluid are constant $\mathbf{f}^F=(f_1^F,f_2^F)^T\in\mathbb{R}^2$ and we consider $\widehat{\mathbf{f}}^F(\alpha)\in\widehat{W}'$ defined by

$$\left\langle\widehat{\mathbf{f}}^F\left(\alpha\right),\widehat{w}\right\rangle=\sum_{i=1}^{2}\int_{\Omega_0^F}\frac{H+u\left(\widehat{x}_1\right)}{H}f_i^F\widehat{w}_i\,d\widehat{\mathbf{x}},\quad\forall\widehat{\mathbf{w}}\in\widehat{W}.$$

We remark that the displacement u which appears in the coefficients depends on α.

The problem: find $\widehat{\mathbf{v}}\in\left(H^1\left(\Omega_0^F\right)\right)^2$, $\widehat{\mathbf{v}}|_\Sigma=\mathbf{g}$, $\widehat{\mathbf{v}}|_{\Gamma_0}=0$, $\widehat{p}\in\widehat{Q}$ *such that*

$$\left\{\begin{array}{lcll}\widehat{a}_F\left(\alpha,\widehat{\mathbf{v}},\widehat{\mathbf{w}}\right)+\widehat{b}_F\left(\alpha,\widehat{\mathbf{w}},\widehat{p}\right) &=& \left\langle\widehat{\mathbf{f}}^F\left(\alpha\right),\widehat{\mathbf{w}}\right\rangle, & \forall\widehat{\mathbf{w}}\in\widehat{W}\\ \widehat{b}_F\left(\alpha,\widehat{\mathbf{v}},\widehat{q}\right) &=& 0, & \forall\widehat{q}\in\widehat{Q}\end{array}\right.\tag{21}$$

has a unique solution.

The problem (21) is obtained from (15) and conversely by using the one-to-one transformations T_u and T_u^{-1}. The equivalence of (21) and (15) follows from the transport theorems in continuum mechanics (see [23]), the chain rule and basic results for Sobolev spaces (see [24]). The conclusion of this proposition is a consequence of the existence and uniqueness of (15).

Let $\widehat{\mathbf{v}}$, $\widehat{\mathbf{w}}$ *be given in* $\left(H^1\left(\Omega_0^F\right)\right)^2$ *and* $\widehat{q}$ *in* $\widehat{Q}$. *Then functions from* $\mathbb{R}^m$ *to* $\mathbb{R}$ *defined by*

$$\begin{array}{lcl}\alpha &\longmapsto& \widehat{a}_F\left(\alpha,\widehat{\mathbf{v}},\widehat{\mathbf{w}}\right)\\ \alpha &\longmapsto& \widehat{b}_F\left(\alpha,\widehat{\mathbf{w}},\widehat{q}\right)\\ \alpha &\longmapsto& \left\langle\widehat{\mathbf{f}}^F\left(\alpha\right),\widehat{\mathbf{w}}\right\rangle\end{array}$$

are differentiable and the partial derivatives have the forms:

$$\begin{aligned}
\frac{\partial\widehat{a}_F}{\partial\alpha_k}\left(\alpha,\widehat{\mathbf{v}},\widehat{\mathbf{w}}\right) &= \sum_{i=1}^{2}\int_{\Omega_0^F}\left(\frac{u_k\left(\widehat{x}_1\right)}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_1}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_1}-\frac{u_k'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_1}\right)d\widehat{\mathbf{x}}\\
&+\sum_{i=1}^{2}\int_{\Omega_0^F}\left(-\frac{u_k'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_1}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}\right)d\widehat{\mathbf{x}}\\
&+\sum_{i=1}^{2}\int_{\Omega_0^F}\left(\frac{2u_k'\left(\widehat{x}_1\right)u'\left(\widehat{x}_1\right)\left(\widehat{x}_2\right)^2}{H\left(H+u\left(\widehat{x}_1\right)\right)}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}\right)d\widehat{\mathbf{x}}\\
&+\sum_{i=1}^{2}\int_{\Omega_0^F}\left(\frac{-u_k\left(\widehat{x}_1\right)\left(H^2+\left(u'\left(\widehat{x}_1\right)\widehat{x}_2\right)^2\right)}{H\left(H+u\left(\widehat{x}_1\right)\right)^2}\frac{\partial\widehat{v}_i}{\partial\widehat{x}_2}\frac{\partial\widehat{w}_i}{\partial\widehat{x}_2}\right)d\widehat{\mathbf{x}}\\
\frac{\partial\widehat{b}_F}{\partial\alpha_k}\left(\alpha,\widehat{\mathbf{w}},\widehat{q}\right) &= -\int_{\Omega_0^F}\left(\frac{u_k\left(\widehat{x}_1\right)}{H}\frac{\partial\widehat{w}_1}{\partial\widehat{x}_1}-\frac{u_k'\left(\widehat{x}_1\right)\widehat{x}_2}{H}\frac{\partial\widehat{w}_1}{\partial\widehat{x}_2}\right)\widehat{q}\,d\widehat{\mathbf{x}}\\
\frac{\partial}{\partial\alpha_k}\left\langle\widehat{\mathbf{f}}^F\left(\alpha\right),\widehat{\mathbf{w}}\right\rangle &= \sum_{i=1}^{2}\int_{\Omega_0^F}\frac{u_k\left(\widehat{x}_1\right)}{H}f_i^F\widehat{w}_i\,d\widehat{\mathbf{x}}.
\end{aligned}$$

This above result is a consequence of the differentiability of integrals with respect to parameters. In our case the parameter is α. Applying the Implicit Function Theorem, we obtain the following result.

The applications $\alpha \in \mathbb{R}^m \mapsto \widehat{\mathbf{v}} \in \left(H^1\left(\Omega_0^F\right)\right)^2$ and $\alpha \in \mathbb{R}^m \mapsto \widehat{p} \in \widehat{Q}$ are differentiables and the partial derivatives $\frac{\partial \widehat{\mathbf{v}}}{\partial \alpha_k} \in \widehat{W}$ and $\frac{\partial \widehat{p}}{\partial \alpha_k} \in \widehat{Q}$ verify

$$\left\{ \begin{array}{lcl} \widehat{a}_F\left(\alpha, \frac{\partial \widehat{\mathbf{v}}}{\partial \alpha_k}, \widehat{\mathbf{w}}\right) + \widehat{b}_F\left(\alpha, \widehat{\mathbf{w}}, \frac{\partial \widehat{p}}{\partial \alpha_k}\right) & = & \frac{\partial}{\partial \alpha_k}\left\langle \widehat{\mathbf{f}}^F(\alpha), \widehat{\mathbf{w}}\right\rangle - \frac{\partial \widehat{a}_F}{\partial \alpha_k}(\alpha, \widehat{\mathbf{v}}, \widehat{\mathbf{w}}) - \frac{\partial \widehat{b}_F}{\partial \alpha_k}(\alpha, \widehat{\mathbf{w}}, \widehat{p}), \\ \widehat{b}_F\left(\alpha, \frac{\partial \widehat{\mathbf{v}}}{\partial \alpha_k}, \widehat{q}\right) & = & -\frac{\partial \widehat{b}_F}{\partial \alpha_k}(\alpha, \widehat{\mathbf{v}}, \widehat{q}) \end{array} \right. \tag{22}$$

for all $\widehat{\mathbf{w}}$ in $\widehat{W}$ and for all $\widehat{q}$ in $\widehat{Q}$.

7.3 Sensitivity of the cost function

If $p_0 \in H^1\left(\Omega_u^F\right)$ such that $\int_0^L p_0\left(x_1, H + u(x_1)\right) dx_1 = 0$, then $\int_0^L \widehat{p}_0\left(x_1, H\right) dx_1 = 0$, where $\widehat{p}_0 = p_0 \circ T_u$. Also, we have $\int_0^L \frac{\partial \widehat{p}_0}{\partial \alpha_k}\left(x_1, H\right) dx_1 = 0$.

The application $\alpha \in \mathbb{R}^m \mapsto J(\alpha)$ is differentiable and the partial derivatives $\frac{\partial J}{\partial \alpha_k}(\alpha)$ have the forms:

$$2\int_0^L \left(\phi_k(x_1) - \int_0^L \frac{\phi_k}{L} dx_1 - \frac{\partial \widehat{p}_0}{\partial \alpha_k}(x_1, H)\right)\left(\sum_{i=1}^m \alpha_i\left(\phi_i(x_1) - \int_0^L \frac{\phi_i}{L} dx_1\right) - \widehat{p}_0\left(x_1, H\right)\right) dx_1 \tag{23}$$

8 Numerical results

We are interested in simulating the blood flow through medium vessels (arteries). The computation has been made in a domain of length $L = 3\ cm$ and height $H = 0.5\ cm$ which represents a half width of the vessel. In this case, the fluid is the blood and the structure is the wall of the vessel.

The numerical values of the following physical parameters have been taken from [1]. The viscosity of the blood was taken to be $\mu = 0.035\ \frac{g}{cm \cdot s}$, its density $\rho^F = 1\ \frac{g}{cm^3}$. The thickness of the vessel is $h = 0.1\ cm$, the Young modulus $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the density $\rho^S = 1.1\ \frac{g}{cm^3}$.

The gravitational acceleration is $g_0 = 981\ \frac{cm}{s^2}$ and the averaged volume force of the structure is $f^S(x_1) = -g_0 \rho^S h$.

On the rigid boundary, we impose the following boundary conditions:

$$\begin{array}{rcl} v_1(x_1, x_2) & = & \left\{ \begin{array}{ll} \left(1 - \frac{x_2^2}{H^2}\right) V_0, & (x_1, x_2) \in \Sigma_1 \cup \Sigma_3 \\ V_0, & (x_1, x_2) \in \Sigma_2 \end{array} \right. \\ v_2(x_1, x_2) & = & 0, \quad (x_1, x_2) \in \Sigma \end{array}$$

where $V_0 = 30\ \frac{cm}{s}$ (see [25]). The volume force in fluid is $\mathbf{f}^F = (0, -g_0 \rho^F)^T$.

The numerical tests have been produced using *freefem++ v1.27* (see [26]).

For the fluid we have used the Mixed Finite Element Method, $P2$ Lagrange triangles for the velocity and $P1$ for the pressure.

8.1 Optimization without using the derivative

Numerical test 1.

We use the same notations as in the previous sections, in particular m and ϕ_i refer to the equation (16). We set $m = 1$ and $\phi_1(x_1) = x_1 - L/2$. In this case $c_0 = g_0 \rho^S h$, $u_0 = 0$, $c_1 = 0$ and

$$u_1(x_1) = \frac{x_1^2 (L - x_1)^2 (x_1 - L/2)}{240D}, \quad u(x_1) = \alpha_1 u_1(x_1).$$

We remark that the displacement of the interface is computed exactly.

We have evaluated the cost function for equidistant points of step length 0.5 in the interval $[-20, 5]$. For each α_1, we generate a mesh for Ω_u^F, where the displacement u depends on α_1. A typical mesh of 198 triangles and 128 vertices is shown below.

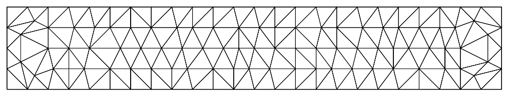

Figure 2: A typical mesh

The condition (4) was not violated. Then, we solve the Stokes equations (15) on this mesh.

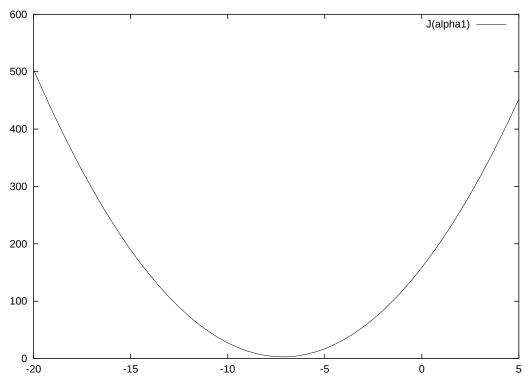

Figure 3: The cost function

The graph $\alpha_1 \to J(\alpha_1)$ seems to be strictly convex, consequently the optimal control is unique (see Figure 3). The cost function has the value $J = 158.76$ for $\alpha_1 = 0$. The minimal value of the cost function $J = 3.04$ was obtained for $\alpha_1 = -7$.

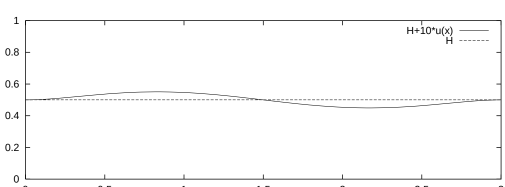

Figure 4: The displacement of the vessel magnified by 10

The displacement of the vessel is very small, so the behavior of the blood flow is like the Poiseuille flow.

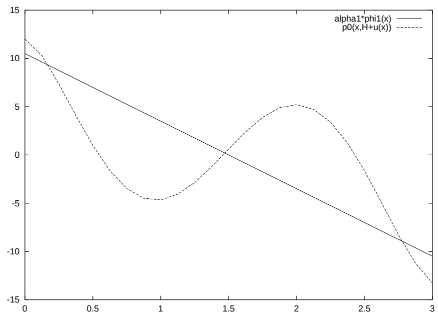

Figure 5: The optimal control $\alpha_1\phi_1(x_1)$ and the optimal observation $p_0(x_1, H+u(x_1))$

The optimal control is -7 and the pressure on the interface can be approached by $-7(x_1 - L/2) + g_0\rho^S h$. The pressure difference between the outflow (right) and inflow (left) is $-7L$.

If we take the averaged volume forces in the vessel of the form $f^S(x_1) = \frac{2\mu V_0}{H^2}x_1 - \rho^S g_0\, h$ we obtain the Poiseuille flow for the blood. The pressure on the interface in this case is $p(x_1, H) = -\frac{2\mu V_0}{H^2}x_1 + g_0\rho^S h$ where $-\frac{2\mu V_0}{H^2} = -8.4$ and the pressure difference between the outflow and inflow is $-8.4L$, so there is a lose of the pressure. The displacement of the interface is consequent: the shape of the vessel is inflow at the left and outflow at the right (see Figure 4). In Figure 5 we observe the difference between the optimal control and the optimal observation. In the fixed point approach, the two graphs must be identical.

If the condition (4) is violated, we have

$$\inf_{x_1\in[0,L]} \{H + u(x_1)\} = 0$$

and we say that the vessel is collapsed. Numerical results for this case are presented in [27].

8.2 The BFGS algorithm

The BFGS algorithm is a quasi-Newton iterative method for solving unconstrained optimization problem $\inf\{J(\alpha);\ \alpha\in\mathbb{R}^m\}$.

Step 0 Choose a starting point $\alpha^0 \in \mathbb{R}^m$, an $m\times m$ symmetric positive matrix H_0 and a positive scalar ϵ. Set $k = 0$.

Step 1 Compute $\nabla J(\alpha^k)$.

Step 2 If $\left\|\nabla J(\alpha^k)\right\| < \epsilon$ stop.

Step 3 Set $\mathbf{d}^k = -H_k\nabla J(\alpha^k)$.

Step 4 Determine $\alpha^{k+1} = \alpha^k + \theta_k\mathbf{d}^k$, $\theta_k > 0$ by means of an approximate minimization

$$J(\alpha^{k+1}) \approx \min_{\theta\geq 0} J(\alpha^k + \theta\mathbf{d}^k).$$

Step 5 Compute $\delta_k = \alpha^{k+1} - \alpha^k$.

Step 6 Compute $\nabla J(\alpha^{k+1})$ and $\gamma_k = \nabla J(\alpha^{k+1}) - \nabla J(\alpha^k)$.

Step 7 Compute

$$H_{k+1} = H_k + \left(1 + \frac{\gamma_k^T H_k\gamma_k}{\delta_k^T\gamma_k}\right)\frac{\delta_k\delta_k^T}{\delta_k^T\gamma_k} - \frac{\delta_k\gamma_k^T H_k + H_k\gamma_k\delta_k^T}{\delta_k^T\gamma_k}$$

Step 8 Update $k = k+1$ and go to the **Step 2**.

For the inaccurate line search at the **Step 4**, the methods of Goldstein and Armijo were used. If we denote by $g : [0, \infty) \to \mathbb{R}$ the function $g(\theta) = J(\alpha^k + \theta \mathbf{d}^k)$, we determine $\theta_k > 0$ such that

$$g(0) + (1-\lambda)\,\theta_k g'(0) \le g(\theta_k) \le g(0) + \lambda \theta_k g'(0) \tag{24}$$

where $\lambda \in (0, 1/2)$.

In the BGSF algorithm, we have used (21) which is the ALE version of the Stokes equations in the reference domain in order to compute the cost function and we have used (22) and (23) in order to compute $\nabla J(\alpha)$.

Remark 2 *In order to compute $\nabla J(\alpha)$ by (22) and (23), we have to solve m linear systems which have the same matrix. The linear systems were solved by LU decomposition. We observe that (21) and (22) have the same left-hand side, so when we compute $\nabla J(\alpha)$ we can use the same LU decomposition obtained computing $J(\alpha)$ by (21).*

We could compute $\nabla J(\alpha)$ by the Finite Differences Method

$$\frac{\partial J}{\partial \alpha_k}(\alpha) \approx \frac{J(\alpha + \Delta\alpha_k \mathbf{e_k}) - J(\alpha)}{\Delta \alpha_k} \tag{25}$$

where $\mathbf{e_k}$ is the k-th vector of the canonical base of $\mathbb{R}^m$ and $\Delta\alpha_k > 0$ is the grid spacing. In this case, the cost function J need to be evaluated in each $\alpha + \Delta\alpha_k \mathbf{e_k}$, $k = 1, \ldots, m$. We have to solve m linear systems obtained from (21), but the matrices are different, so using the analytic formula of the gradient (22) is more advantageous.

Numerical test 2.

We have performed the numerical test in the case $m = 1$ and $\phi_1(x_1) = x_1 - L/2$.

In the table below, we show the gradient of the cost function computed by (22) and (23), respectively by the Finite Differences Method (25) with $\Delta\alpha_1 = 0.5$, which proves the validity of the analytic formula.

α_1	$\nabla J(\alpha_1)$ using (22) and (23)	$\nabla J(\alpha_1)$ using Finite Differences (25)
-20	-77.88	-76.50
-15	-47.55	-46.09
-10	-17.22	-15.70
-5	13.13	14.63
0	43.49	45.03
5	73.87	72.40

The starting point for the BFGS algorithm is $\alpha_1 = 0$ and the stopping criteria is $\|\nabla \mathcal{J}\|_\infty \le 10^{-6}$.

Iterations	α_1	$J(\alpha_1)$	$\|\nabla J(\alpha_1)\|_\infty$
0	0	158.70	43.49
1	-43.49	4003.66	-220.03
2	-7.17907	2.95985	-0.100582
3	-7.16247	2.95899	0.000232724
4	-7.1625	2.95899	-2.53259e-10

The condition (4) was not violated. The minimal value of the cost function $J = 2.95899$ was obtained for $\alpha_1 = -7.1625$, after 5 iterations. The line search algorithm for the approximate minimization at the Step 4 was not activated, we take $\theta_K = 1$. The computed displacements of the vessel are almost the same as in the Figure 4. If we activate the line search algorithm and we set to 3 the maximal number of evaluation of the cost function at the Step 4, we obtain $\alpha_1^0 = 0$, $\alpha_1^1 = -7.17207$, $\alpha_1^2 = -7.16251$, $\alpha_1^3 = -7.16249$, $\alpha_1^4 = -7.1625$.

Numerical test 3.

We take $m = 4$. Let $\xi_i = (i-1)L/(m-1)$ for $1 \le i \le m$ be an uniform grid of $[0, L]$. For each $i = 1, \ldots, m$, there exists a unique ϕ_i polynomial function of degree 3, such that $\phi_i(\xi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker's symbol. The functions ϕ_i are not necessary the same as the trace on the interface of the pressure finite element functions. Other choice for ϕ_i could be the vibration modes of the beam equations.

Let u_i, c_i be the solutions of (20). We have computed u_i, c_i exactly, using the software *Mathematica*. The displacements u_i are polynomial functions of degree 7.

The fluid equations were solved in the reference mesh shown in Figure 2.

The starting point for the BFGS algorithm is $\alpha = 0$ and the stopping criteria is $\|\nabla \mathcal{J}\|_\infty \le 10^{-6}$. The analytic formula of the gradient was employed.

Iterations	J	$\|\nabla J\|_\infty$
0	158.70	21.29
1	42.88	3.51
2	20.39	2.38
3	0.168155	0.30
4	0.165842	0.008
5	0.165653	2.5e-7

Five iterations are required to achieve $\|\nabla \mathcal{J}\|_\infty \le 10^{-6}$ and the obtained discrete optimal control is

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (13.2723413,\ 2.89419278,\ -2.704038443,\ -13.46249563).$$

The optimal value of the cost function for $m = 4$ is $J = 0.165653$ which is less than $J = 2.95899$ in the case $m = 1$.

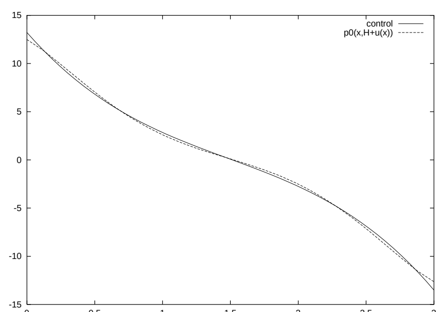

Figure 6: The optimal control function $\sum_{i=1}^m \alpha_i \left(\phi_i(x_1) - \frac{1}{L}\int_0^L \phi_i(x_1)dx_1\right)$ and the optimal observation $p_0(x_1, H + u(x_1))$

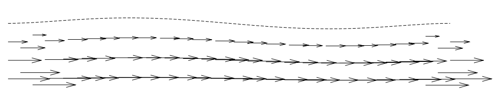

Figure 7: The displacement $[cm]$ of the vessel magnified by a factor 20 and the velocity $[cm/s]$ reduced by a factor 100

The displacement of the vessel is very small, it is less than 0.04 cm. The computed velocity distribution is similar to a Poiseuille flow (see Figure 7).

9 Conclusions

Using the Least Squares Method and the Arbitrary Lagrangian Eulerian coordinates, a two dimensional steady fluid structure interaction problem was transformed into an optimal control problem.

The BFGS algorithm has given satisfactory numerical results even when a reduced number of discrete controls were used. The analytic formula of the gradient was employed. Computational results reveal that the displacement of the interface is very small when the velocity profile is parabolic at the inflow and outflow.

We have obtained a smaller optimal value by increasing the number of the controls and by changing the shape of the control functions.

In a forthcoming paper, the techniques used here will be adapted to the unsteady fluid-structure interaction problems. The vibration modes of the structure will be the control shape functions.

Acknowledgments

The author is very grateful to the anonymous referees whose suggestions and comments have contributed to improve the content of this paper.

References

[1] L. Formaggia, J. F. Gerbeau, F. Nobile and A. Quarteroni, On the coupling of 3D and 1D Navier-Stokes equations for flow problems in compliant vessels, *Comput. Methods Appl. Mech. Engrg.* **191**(6-7) 561–582 (2001).

[2] J. F. Gerbeau and M. Vidrascu, A quasi-Newton algorithm on a reduced model for fluid - structure interaction problems in blood flows, *M2AN Math. Model. Numer. Anal.* **37**(4) 631–647 (2003).

[3] H. Chen and T. Sheu, Finite-element simulation of incompressible fluid flow in an elastic vessel, *Int. J. Numer. Meth. Fluids* **42** 131–146 (2003).

[4] C. Farhat, Ph. Geuzaine and C. Grandmont, The discrete geometric conservation law and the nonlinear stability of ALE schemes for the solution of flow problems on moving grids, *J. Comput. Phys.* **174**(2) 669–694 (2001).

[5] C. Grandmont, Existence for a three-dimensional steady state fluid-structure interaction problem, *J. Math. Fluid Mech.* **4**(1) 76–94 (2002).

[6] G. Bayada, M. Chambat, B. Cid and C. Vazquez, On the existence of solution for a non-homogeneus Stokes-rod coupled problem, preprint Equipe d'Analyse Numérique Lyon-Saint Etienne (UMR 5585), No. 362 (2003).

[7] C. Grandmont and Y. Maday, Existence for an unsteady fluid-structure interaction problem, *M2AN Math. Model. Numer. Anal.* **34**(3) 609–636 (2000).

[8] B. Desjardins, M. Esteban, C. Grandmont and P. Le Tallec, Weak solutions for a fluid-elastic structure interaction model, *Rev. Mat. Complut.* **14**(2) 523–538 (2001).

[9] H. Beirao da Veiga, On the existence of strong solution to a coupled fluid structure evolution problem, *J. Math. Fluid Mech.* **6**, 21–52 (2004).

[10] M. A. Fernandez and M. Moubachir, Sensitivity analysis for an incompressible aeroelastic system, *Math. Models Methods Appl. Sci.* **12**(8) 1109–1130 (2002).

[11] G. Bayada, J. Durany and C. Vázquez, Existence of a Solution for a Lubrication Problem in Elastic Journal Bearings with Thin Bearing, *Math. Meth. Appl. Sci.*, **18** 455–466 (1995).

[12] C. Murea and Y. Maday, Existence of an Optimal Control for a Nonlinear Fluid-Cable Interaction Problem, Rapport de recherche CEMRACS 96, C.I.R.M. Luminy, France, (1996).

[13] C. Murea and C. Vázquez, Sensitivity and approximation of fluid-structure coupled by virtual control method, submitted.

[14] R.M. Lewis, A nonlinear programming perspective on sensitivity calculations for systems governed by state equations, NASA-ICASE Report no. 12, (1997).

[15] C. Murea, Optimal control approach for the fluid-structure interaction problems, In *Proceedings of the Fourth European Conference on Elliptic and Parabolic Problems, Rolduc and Gaeta*, 2001, (Edited by J. Bemelmans et al), pp. 442–450, World Scientific Publishing Co. Pte. Ltd., (2002).

[16] P. Le Tallec and J. Mouro, Fluid-structure interaction with large structural displacements, *Comput. Methods Appl. Mech. Engrg.* **190**(24-25) 3039–3067 (2001).

[17] F. Nobile, Numerical approximation of fluid-structure interaction problems with application to haemodynamics, PhD, Ecole Polytechnique Féderale de Lausanne, Switzerland, (2001).

[18] S. Deparis, M.A. Fernandez and L. Formaggia, Acceleration of a fixed point algorithm for fluid-structure interaction using transpiration conditions, *M2AN Math. Model. Numer. Anal.* **37**(4) 601–616 (2003).

[19] J. Steindorf and H.G. Matthies, Partioned but strongly coupled iteration schemes for nonlinear fluid-structure interaction, to appear.

[20] M. Bernadou, Formulation variationnelle, approximation et implementation de problemes de barres et de poutres bi- et tri-dimensionnelles. Partie B : barres et poutres bidimensionnelles, Rapport Techique no 86, INRIA, (1987).

[21] A. Osses, A rotated multiplier applied to the controllability of waves, elasticity and tangential Stokes control, *SIAM J. Control Optim.* **40**(3) 777–800 (2001).

[22] V. Girault and P.-A. Raviart, *Finite element methods for Navier-Stokes equations. Theory and algorithms*, Springer Series in Computational Mathematics, Springer-Verlag, Berlin, (1986).

[23] M. E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, (1981).

[24] R. Adams, *Sobolev spaces*, Academic Press, (1975).

[25] K. Schmidt-Nielsen, *Physiologie animale. Adaptation et milieux de vie*, Dunod, (1998).

[26] F. Hecht and O. Pironneau, A finite element software for PDE: freefem++, `http://www-rocq.inria.fr/Frederic.Hecht` [May 2003]

[27] M. Thiriet, A Three dimensional Numerical Method of a Critical Flow in a Collapsed Tube, Rapport de Recherche no 3867, INRIA, (2000).