Numerical experiment for stabilization of the heat equation by Dirichlet boundary control *

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Abstract

We use the boundary feedback control introduced in the paper [V. Barbu, Boundary stabilization of equilibrium solutions to parabolic equations, IEEE Trans. Automat. Control, (2013)], in order to stabilize an unstable heat equation in two dimensions. We propose two numerical algorithms. The feedback boundary condition is treated explicitly in the first algorithm. At each time step, only one linear system is solved. The second algorithm performs at each time step some sub iterations, in order to treat the feedback boundary condition implicitly. The second algorithm can stabilize some problems where the first algorithm fails.

keywords. boundary feedback control, unstable heat equation in 2D, numerical method, auto-ignition, ignition control

1 Introduction

In [1] a new technique was developed for the construction of stabilizable feedback boundary controllers for parabolic equations.

The stabilizing control is expressed using the system of eigenfunctions corresponding to unstable eigenvalues of the linear system associated to the original one by linearization with respect to the steady-state.

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The present paper aims to illustrate through numerical simulations the effectiveness of the method introduced in [1]. We will follow closely [1] in the presentation of the problem.

Consider the mixed problem for the parabolic equation

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \Delta y + f(x,y), \text{ in } (0,\infty)\times\Omega && (1)\\
y &= u, \text{ on } (0,\infty)\times\Gamma_1 && (2)\\
\frac{\partial y}{\partial n} &= 0, \text{ on } (0,\infty)\times\Gamma_2 && (3)\\
y(0,x) &= y_0(x), \text{ in } \Omega, && (4)
\end{aligned}$$

Ω an open and bounded domain in $\mathbb{R}^d$ with smooth boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 non overlapping connected parts of $\partial\Omega$. The solution y depends on $(t,x) \in [0,\infty)\times\overline{\Omega}$, $x = (x_1,\dots,x_d)$.

Let y_e be a solution for

$$\Delta y_e + f(x,y_e) = 0, \text{ in } \Omega, \quad \frac{\partial y_e}{\partial n} = 0 \text{ on } \Gamma_2.$$

With $\tilde{y} = y - y_e$ a translation to zero, the system (1)-(4) becomes

$$\begin{aligned}
\frac{\partial \tilde{y}}{\partial t} &= \Delta \tilde{y} + f(x,\tilde{y}+y_e) - f(x,y_e), \text{ in } (0,\infty)\times\Omega && (5)\\
\tilde{y} &= u - y_e, \text{ on } (0,\infty)\times\Gamma_1 && (6)\\
\frac{\partial \tilde{y}}{\partial n} &= 0, \text{ on } (0,\infty)\times\Gamma_2 && (7)\\
\tilde{y}(0,x) &= y_0(x) - y_e = \tilde{y}_0(x), \text{ in } \Omega. && (8)
\end{aligned}$$

The problem is now to synthesize a feedback controller $u = F(\tilde{y})$ such that, for all $\tilde{y}_0$ in a neighborhood of the origin in $L^2(\Omega)$, the solution to the closed loop system satisfies

$$\int_\Omega |\tilde{y}(t,x)|^2\,dx \le Ce^{-\gamma t}\int_\Omega |\tilde{y}_0(x)|^2\,dx, \quad t \ge 0. \qquad (9)$$

The strategy proposed in [1] is to first find a stabilizing controller $v = F(y)$ for the linearization of (5)-(8) in zero and to use it afterwards to stabilize the zero solution of (1)-(4) thus the solution y_e.

The linearization of (5)-(8) in zero gives the problem

$$\begin{aligned}
\frac{\partial y}{\partial t} &= \Delta y + \frac{\partial f}{\partial y}(x,y_e)y, \text{ in } (0,\infty)\times\Omega && (10)\\
y &= v, \text{ on } (0,\infty)\times\Gamma_1 && (11)\\
\frac{\partial y}{\partial n} &= 0, \text{ on } (0,\infty)\times\Gamma_2 && (12)\\
y(0,x) &= \tilde{y}_0(x), \text{ in } \Omega. && (13)
\end{aligned}$$

It will be assumed that

$$f, \frac{\partial f}{\partial y} \in \mathcal{C}\left(\overline{\Omega} \times \mathbb{R}\right).$$

Consider $L : \mathcal{D}(L) \to L^2(\Omega)$,

$$Ly = \Delta y + \frac{\partial f}{\partial y}(x, y_e)y, \quad y \in \mathcal{D}(L)$$

$$\mathcal{D}(L) = \left\{ y \in H^2(\Omega);\ y = 0 \text{ on } \Gamma_1,\ \frac{\partial y}{\partial n} = 0 \text{ on } \Gamma_2 \right\}.$$

Then the resolvent of L is compact and

$$\langle L\, y, y \rangle \leq - \|\nabla y\|^2_{L^2(\Omega)} - C\, \|y\|^2_{L^2(\Omega)}.$$

It follows that $-L$ has a countable set of real eigenvalues λ_j of finite multiplicity with the corresponding eigenfunctions φ_j, $-L\varphi_j = \lambda_j \varphi_j$. We assume that $\lambda_1 \leq \lambda_2 \leq \ldots$.

There exits a finite number $N \in \mathbb{N}$ such that $\lambda_j < 0$ for $j = 1, \ldots, N$ and $\lambda_j > 0$ for $j = N+1, \ldots$. Then $\frac{\partial \varphi_j}{\partial n} = 0$ on Γ_2 and the following standard hypothesis is assumed to hold true:

$$\frac{\partial \varphi_j}{\partial n} : \Gamma_1 \to \mathbb{R},\ j = 1, \ldots, N \text{ are linearly independent.} \tag{14}$$

The following controller is defined in [1]

$$v(t,x) = \eta \sum_{j=1}^{N} \mu_j \left(\int_\Omega y(t,x)\varphi_j(x)\, dx \right) \phi_j(x), \text{ on } (0,\infty) \times \Gamma_1 \tag{15}$$

where

$$\mu_j = \frac{k + \lambda_j}{k + \lambda_j - \eta},\ j = 1, \ldots, N,$$

$$\phi_j(x) = \sum_{\ell=1}^{N} a_{j\ell} \frac{\partial \varphi_\ell}{\partial n}(x),\ \forall x \in \Gamma_1,$$

$\eta, k > 0$ are chosen sufficiently large such that

$$\lambda_j + \eta + \frac{2\eta^2}{\lambda_j + k - 2\eta} \geq \gamma_0 > 0,\ j = 1, \ldots, N \tag{16}$$

and $(a_{ij})_{1 \leq i,j \leq N}$ is given by

$$(a_{ij})_{1 \leq i,j \leq N} = M^{-1} \quad \text{and} \quad M = \left(\int_{\Gamma_1} \frac{\partial \varphi_i}{\partial n} \frac{\partial \varphi_j}{\partial n} ds \right)_{1 \leq i,j \leq N}.$$

In the present paper, the parameters η and k are chosen as follows: $\eta > -\lambda_1$ and $k = 4\eta$. Then, the inequality (16) is verified.

It is proved in [1], that, if the condition (16) holds, then the feedback controller (15) stabilizes exponentially (10)-(13), more precisely,

$$\|y(t)\|_{L^2(\Omega)} \leq Ce^{-\gamma t} \|y_0\|_{L^2(\Omega)} .$$

where $\gamma = \min\{\gamma_0, \lambda_{N+1}\} > 0$.

The backstepping approach is an efficient method for boundary stabilization of one-dimensional ($d = 1$) unstable heat equations (17)-(20). The stabilization is obtained under the condition $\lambda < 3\pi^2/4$ in [2]. In [3], [4] the stabilization holds with an arbitrary level of instability $\lambda(x)$ and for any exponentially decay rate γ. The case with time dependent coefficient $\lambda(t)$ is studied in [5].

The boundary feedback stabilization for parabolic equation for $d \geq 1$ was established in [6]. The advantage of the boundary feedback controller proposed in [1] is that the construction of the controller is explicit and it is easily implementable.

We use the boundary feedback control, introduced in [1], in order to stabilize an unstable heat equation in two dimensions. We propose two numerical algorithms. The feedback boundary condition is treated explicitly in the first algorithm. At each time step, only one linear system is solved. The second algorithm performs at each time step some sub iterations, in order to treat the feedback boundary condition implicitly. The second algorithm can stabilize some problems where the first algorithm fails.

The results can be applied in the control of thermal systems in which the conductive, convective and advective phenomena are studied. Commonly, heat equation in one-dimension is used as mathematical model for components as fins, long tubes and heat exchangers, [7]. Two-dimensional heat equation will describe thermal processes on bodies with axial symmetry or on surface with small thickness. The temperature in a solid body is a three-dimensional heat equation problem.

The theoretical and numerical results are useful in many engineering applications, including the electric power industry, the automotive industry, the heating, ventilation and air conditioning industry, metallurgical processes of solidification and quenching to improve and optimize thermal processes. Some practical problems are experiencing with process optimization of combustion in order to control and monitories the pollutant emissions and efficiency. For example, controlling the auto-ignition process in petrol engines reduce the exhaust emissions and fuel consumption [8].

The direct measurements of the temperature at the inside surface of combustion chamber or at the surface of a reentry vehicle or the inside surface under fire are difficult and not accurate. For this reason the practical problems regarding thermal processes need to predict the temperature distribution and to have information on boundaries. From this point of view to control the atmospheric reentry and maintain the thermal shield of a space vehicle at a suitable temperature is a big challenge [9].

2 Presentation of unstable heat equation and algorithms

Using the feedback Dirichlet boundary control introduced in [1], we want to stabilize the following parabolic problem:

$$\begin{aligned}
\frac{\partial y}{\partial t}(t,x) - \Delta y(t,x) - \lambda y(t,x) &= 0, \text{ in } (0,\infty)\times\Omega && (17)\\
y(t,x) &= u(t,x), \text{ on } (0,\infty)\times\Gamma_1 && (18)\\
\frac{\partial y}{\partial n}(t,x) &= 0, \text{ on } (0,\infty)\times\Gamma_2 && (19)\\
y(0,x) &= y_0(x), \text{ in } \Omega && (20)
\end{aligned}$$

where λ is a given real number, y_0 the initial condition and u the Dirichlet boundary control.

The eigenvalues problem

$$\begin{aligned}
-\Delta\varphi_j(x) - \lambda\varphi_j(x) &= \lambda_j\varphi_j(x), \text{ in } \Omega && (21)\\
\varphi_j(x) &= 0, \text{ on } \Gamma_1 && (22)\\
\frac{\partial \varphi_j}{\partial n}(x) &= 0, \text{ on } \Gamma_2 && (23)
\end{aligned}$$

has real eigenvalues λ_j and the eigenfunction φ_j, $j = 1, 2, \ldots$

We solve the uncontrolled problem, i.e. (17)-(20) with $u = 0$, using implicit (backward) Euler method. Find y^{n+1} such that

$$\begin{aligned}
\frac{y^{n+1} - y^n}{\Delta t} - \Delta y^{n+1} - \lambda y^{n+1} &= 0, \text{ in } \Omega && (24)\\
y^{n+1} &= 0, \text{ on } \Gamma_1 && (25)\\
\frac{\partial y^{n+1}}{\partial n} &= 0, \text{ on } \Gamma_2 && (26)\\
y^0 &= y_0, \text{ in } \Omega && (27)
\end{aligned}$$

where $y^n(x)$ approaches $y(n\Delta t, x)$ and Δt is the time step.

Algorithm 1. Explicit treatment of the Dirichlet boundary control

Find y^{n+1} verifying (24), (26), (27) with the boundary condition

$$y^{n+1} = u^n, \text{ on } \Gamma_1 \qquad (28)$$

where $u^n(x)$ is obtained by replacing $y(t,x)$ by $y^n(x)$ in the expression (15).

Other possibility is to replace the boundary condition (28) by

$$y^{n+1} = 2u^n - u^{n-1}, \text{ on } \Gamma_1. \tag{29}$$

Algorithm 2. Implicit treatment of the Dirichlet boundary control

At each time step, we have to do:
Step 1. Put $y_0^{n+1} = y^n$, $k = 0$.
Step 2. Find y_{k+1}^{n+1} such that

$$\frac{y_{k+1}^{n+1} - y^n}{\Delta t} - \Delta y_{k+1}^{n+1} - \lambda y_{k+1}^{n+1} = 0, \text{ in } \Omega \tag{30}$$

$$y_{k+1}^{n+1} = \eta \sum_{j=1}^{N} \mu_j \left(\int_\Omega y_k^{n+1} \varphi_j(x)\, dx \right) \phi_j(x), \text{ on } \Gamma_1 \tag{31}$$

$$\frac{\partial y_{k+1}^{n+1}}{\partial n} = 0, \text{ on } \Gamma_2 \tag{32}$$

Step 3. If $\left\| y_{k+1}^{n+1} - y_k^{n+1} \right\|_{L^2(\Omega)} \le tol$ then $y^{n+1} = y_{k+1}^{n+1}$ and stop else $k = k+1$ and go to **Step 2.**

Remark 1 *At each time step, we have to perform some sub iterations $k = 0, 1, \dots$. If we have convergence for $k \to \infty$, then the Dirichlet boundary control is treated implicitly*

$$y^{n+1} = \eta \sum_{j=1}^{N} \mu_j \left(\int_\Omega y^{n+1} \varphi_j(x)\, dx \right) \phi_j(x), \text{ on } \Gamma_1.$$

3 Test 1. Explicit treatment of the control

We set $\Omega = (0,1) \times (0,1)$, $\Gamma_1 = \{1\} \times (0,1)$ and $\Gamma_N = \partial\Omega \setminus \Gamma_2$.

We use the finite element $\mathbb{P}_2$ with a mesh of 594 triangles and 330 vertices.

The eigenvalues problem (21)-(23) has real eigenvalues: $\lambda_1 = 2.46 - \lambda$, $\lambda_2 = 12.33 - \lambda$, $\lambda_3 = 22.20 - \lambda$, $\lambda_4 = 32.07 - \lambda$, $\lambda_5 = 41.94 - \lambda$, etc.

We have used Algorithm 1, with explicit treatment of the Dirichlet boundary control (28). The time step is $\Delta t = 0.005$ and the number of time steps is $NN = 200$.

Case 1. $\lambda = 5$

In this case $\lambda_1 = -2.54 < 0$ and $\lambda_j > 0$ for $j = 2, 3, \dots$ We set $\eta = 3$ and $k = 12$.

Case 2. $\lambda = 15$

In this case $\lambda_1 = -12.54 < 0$, $\lambda_2 = -2.67 < 0$ and $\lambda_j > 0$ for $j = 3, 4, \dots$ We set $\eta = 13$ and $k = 52$.

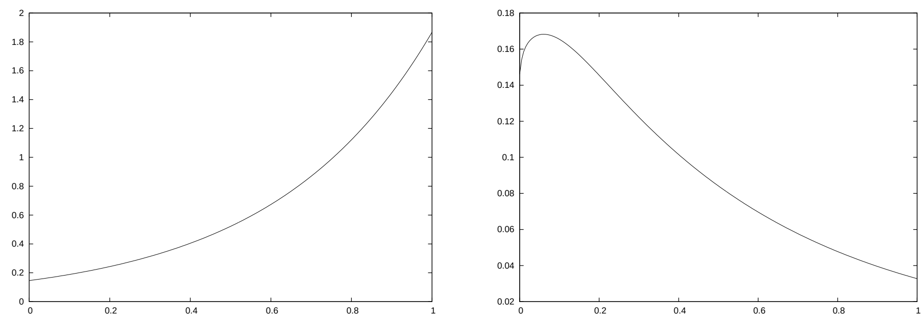

Figure 1: Case 1. The time history of $\|y^n\|_{L^2(\Omega)}$ for initial condition $y_0(x_1, x_2) = 0.2(1 - x_1^2)$. Uncontrolled (left) and controlled (right) cases.

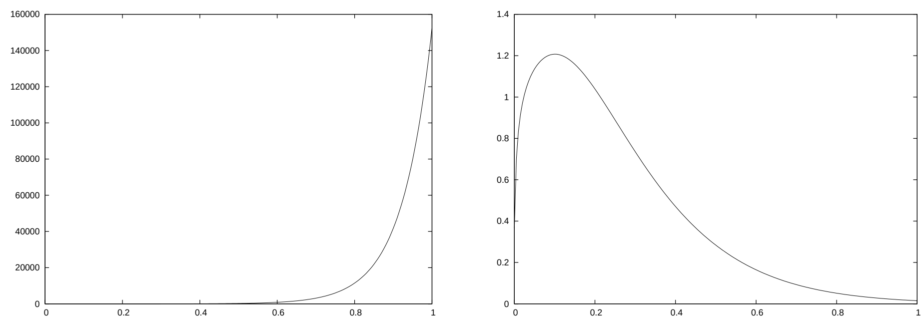

Figure 2: Case 2. The time history of $\|y^n\|_{L^2(\Omega)}$ for initial condition $y_0(x_1, x_2) = 0.5(1 - x_1^2)$. Uncontrolled (left) and controlled (right) cases.

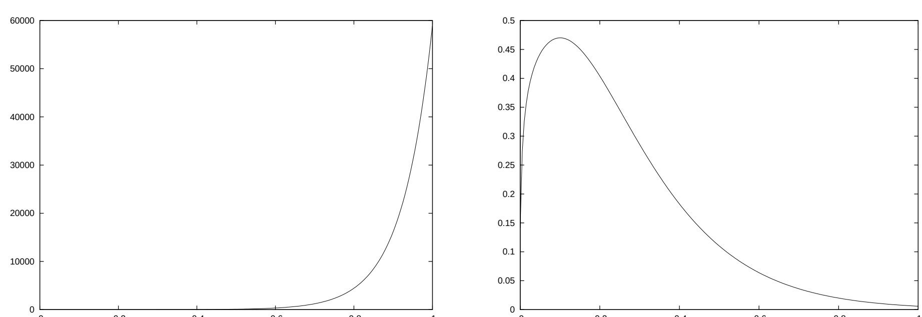

Figure 3: Case 2. The time history of $\|y^n\|_{L^2(\Omega)}$ for initial condition $y_0(x_1, x_2) = 0.2\cos(\frac{\pi}{2}x_1)$. Uncontrolled (left) and controlled (right) cases.

4 Test 2. Implicit treatment of the control

We use the Example 2 from [1]. In this case $\Omega = (0,\pi)\times(0,\pi)$, $\Gamma_1 = \partial\Omega$ and $\Gamma_N = \emptyset$, so the boundary control is defined on the entire border of the domain.

The eigenvalues problem

$$\begin{aligned} -\Delta\varphi_j(x) - \lambda\varphi_j(x) &= \lambda_j\varphi_j(x), \text{ in } \Omega \quad (33)\\ \varphi_j(x) &= 0, \text{ on } \partial\Omega \quad (34)\end{aligned}$$

has the exact eigenvalues: $\lambda_1 = 2-\lambda$, $\lambda_2 = 5-\lambda$, $\lambda_3 = 5-\lambda$, $\lambda_4 = 8-\lambda$, $\lambda_5 = 10-\lambda$, $\lambda_6 = 10-\lambda$, etc.

We have tryed to solve the problem with Algorithm 1, but if the Dirichlet boundary condition is treated explicitly, we are not able to stabilize the problem. We have solved the problem using Algorithm 2 and at the Step 3, we use $tol = 10^{-3}$ for the stopping test.

For $\eta = 50$ and $k = 200$ the inequality (16) is verified.

Case 1. $\lambda = 3$

In this case $\lambda_1 = -1 < 0$ and $\lambda_j > 0$ for $j = 2, 3, \ldots$ The exact eigenfunction is $\varphi_1(x_1,x_2) = \alpha_1 \sin x_1 \sin x_2$. We take $\alpha_1 = \frac{2}{\pi}$ such that $\|\varphi_1\|_{L^2(\Omega)} = 1$.

We obtain that:

$$\frac{\partial\varphi_1}{\partial n}(x_1,x_2) = \begin{cases} -\frac{2}{\pi}\sin x_2, & \text{on } \{0\}\times(0,\pi)\\ -\frac{2}{\pi}\sin x_1, & \text{on } (0,\pi)\times\{0\}\\ -\frac{2}{\pi}\sin x_2, & \text{on } \{\pi\}\times(0,\pi)\\ -\frac{2}{\pi}\sin x_1, & \text{on } (0,\pi)\times\{\pi\}\end{cases}$$

We can compute

$$m_{11} = \int_{\partial\Omega}\left(\frac{\partial\varphi_1}{\partial n}\right)^2 ds = 4\times\left(\frac{2}{\pi}\right)^2\int_0^\pi \sin^2 x_1\, dx_1 = \frac{8}{\pi}$$

then we put $a_{11} = 1/m_{11} = \frac{\pi}{8}$ and finally we set $\phi_1 : \partial\Omega \to \mathbb{R}$, $\phi_1 = a_{11}\frac{\partial\varphi_1}{\partial n}$.

We use the finite element $\mathbb{P}_2$ with a mesh of 242 triangles and 142 vertices. The initial condition is $y_0(x_1,x_2) = 0.2x_1(\pi - x_1)x_2(\pi - x_2)\frac{16}{\pi^4}$. The time step is $\Delta t = 0.01$ and the number of time steps is $NN = 50$.

If the Dirichlet boundary condition is treated implicitly, the feedback control (15) stabilizes the problem, see Figure 4.

Case 2. $\lambda = 7$

In this case $\lambda_1 = -5$, $\lambda_2 = -2$, $\lambda_3 = -2$ and $\lambda_j > 0$ for $j = 4, 5, \ldots$ We use the finite element $\mathbb{P}_2$ with a mesh of 242 triangles and 142 vertices. The initial condition is

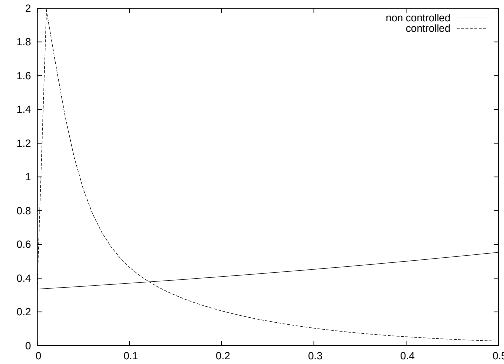

Figure 4: Case 1. Implicit treatment of the control. The time history of $\|y^n\|_{L^2(\Omega)}$.

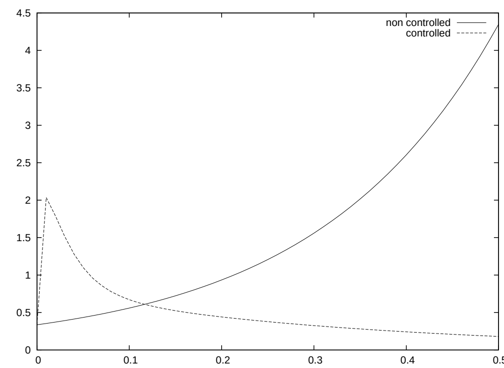

Figure 5: Case 2. Implicit treatment of the control. The time history of $\|y^n\|_{L^2(\Omega)}$.

$y_0(x_1, x_2) = 0.2x_1(\pi - x_1)x_2(\pi - x_2)\frac{16}{\pi^4}$. The time step is $\Delta t = 0.01$ and the number of time steps is $NN = 50$.

The time history of $\|y^n\|_{L^2(\Omega)}$ is shown in Figure 5.

Case 3. $\lambda = 12$

In this case $\lambda_1 = -10$, $\lambda_2 = -7$, $\lambda_3 = -7$, $\lambda_4 = -4$, $\lambda_5 = -2$, $\lambda_6 = -2$ and $\lambda_j > 0$ for $j = 7, 8, \ldots$ We use the finite element $\mathbb{P}_2$ with a mesh of 594 triangles and 330 vertices. The initial condition is $y_0(x_1, x_2) = x_1(\pi - x_1)x_2(\pi - x_2)\frac{16}{\pi^4}$. The time step is $\Delta t = 0.01$ and the number of time steps is $NN = 100$.

The controlled solution at different time instants is presented in Figure 6. In Figure 7, the time history of $\|y^n\|_{L^2(\Omega)}$ is presented.

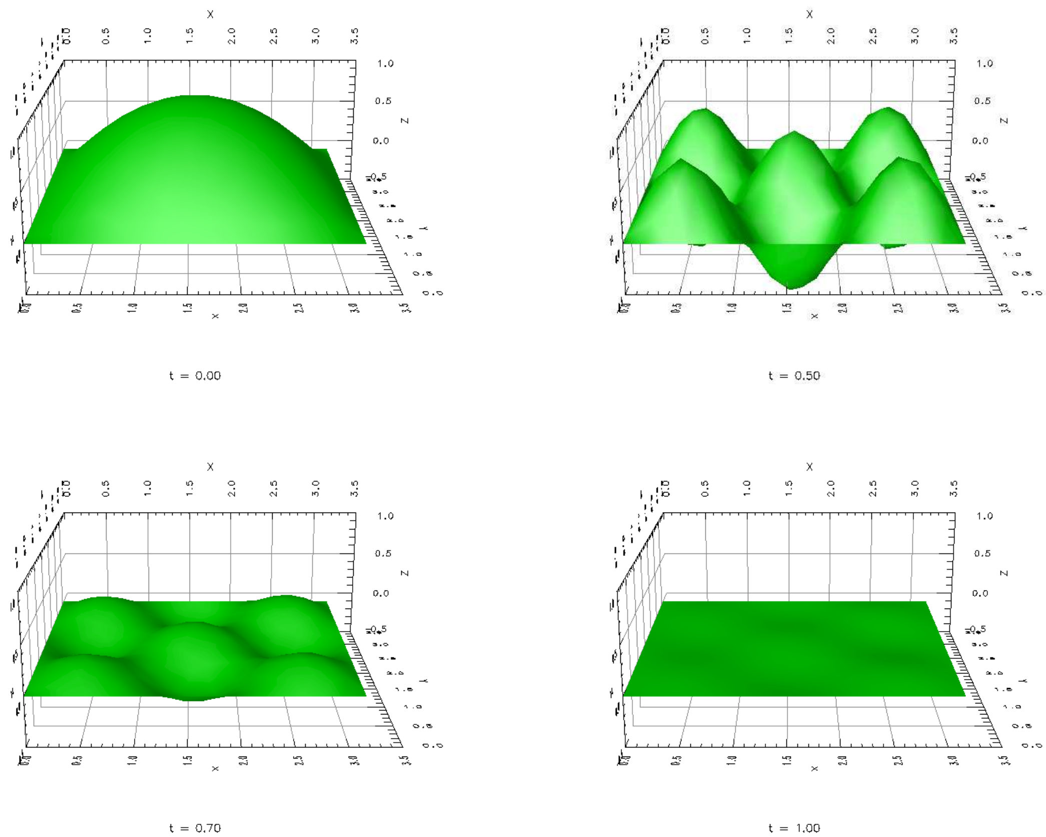

Figure 6: Case 3. The controlled solution at different time instants.

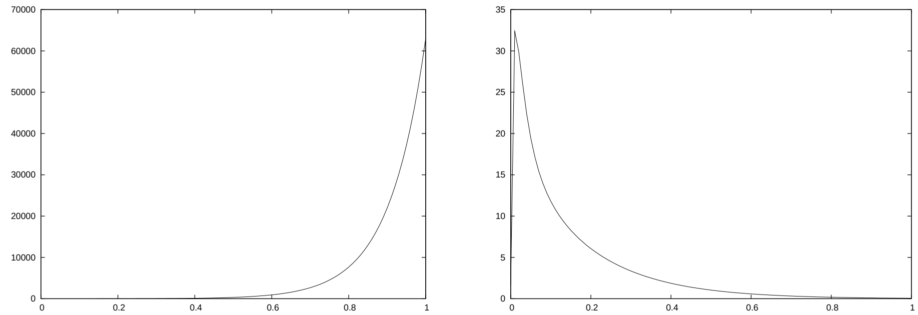

Figure 7: Case 3. Implicit treatment of the control. The time history of $\|y^n\|_{L^2(\Omega)}$. Uncontrolled (left) and controlled (right) cases.

In order to achieve $\left\|y_{k+1}^{n+1} - y_k^{n+1}\right\|_{L^2(\Omega)} \leq 10^{-3}$ at the first time step, 39 iterations are required. This number decreases until the time instant $t = 0.24$. Starting from the time instant $t = 0.25$, only 2 iterations are required in order to achieve the stopping test at the Step 3 of Algorithm 2.

5 Conclusions

Two numerical algorithms are proposed for stabilization of a heat like equation in two dimensions. The treatment of the feedback Dirichlet boundary control is either explicit or implicit. Numerical tests are presented.

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