Numerical control of normal velocity by normal stress for interaction between an incompressible fluid and an elastic curved arch

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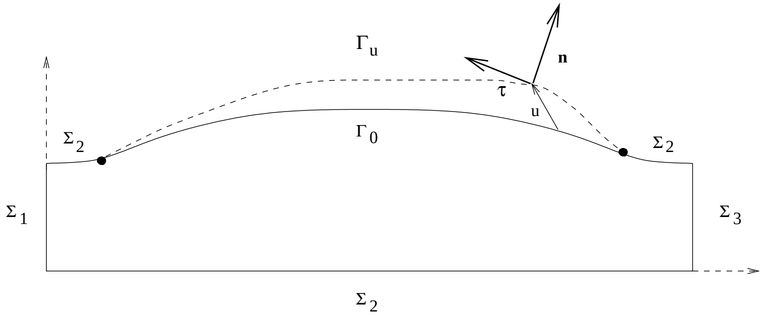
Geometrical model and notations



Notations from thin shell theory

$L > 0$, $\phi^1, \phi^2 : [0, L] \to \mathbb{R}$

$$\Gamma_0 = \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_1 = \phi^1(\xi),\ x_2 = \phi^2(\xi),\ \xi \in [0, L] \right\}$$

The covariant basis $(\mathbf{a_1}, \mathbf{a_3})$ is given by

$$\mathbf{a_1} = \left(\left(\phi^1\right)', \left(\phi^2\right)' \right)^T, \quad \mathbf{a_3} = \frac{1}{\sqrt{a}} \left(\left(\phi^2\right)', -\left(\phi^1\right)' \right)^T$$

and the associated contravariant basis $\left(\mathbf{a^1}, \mathbf{a^3}\right)$ is

$$\mathbf{a^1} = \frac{1}{\sqrt{a}} \mathbf{a_1}, \quad \mathbf{a^3} = \mathbf{a_3}$$

where $a = \left(\left(\phi^1\right)' \right)^2 + \left(\left(\phi^2\right)' \right)^2$.

Weak formulation of the arch problem

$$\mathbf{U} = \left\{ \boldsymbol{\psi} = (\psi_1, \psi_3) \in H_0^1\left(]0, L[\right) \times H_0^2\left(]0, L[\right) \right\}$$

$$a_S\left(\mathbf{u}, \boldsymbol{\psi}\right) = \int_0^L \frac{Ee}{1-\nu^2} \left(\gamma_1^1\left(\mathbf{u}\right) \gamma_1^1\left(\boldsymbol{\psi}\right) + \frac{e^2}{12} \rho_1^1\left(\mathbf{u}\right) \rho_1^1\left(\boldsymbol{\psi}\right) \right) \sqrt{a}\, d\xi$$

Find the displacement $\mathbf{u} = (u_1, u_3) \in U$ of the arch such that for all $\psi \in U$, we have

$$a_S\left(\mathbf{u}, \boldsymbol{\psi}\right) = \int_0^L \left(\lambda^1 \psi_1 + \lambda^3 \psi_3\right) \sqrt{a}\, d\xi + \int_0^L \left(f^{S,1} \psi_1 + f^{S,3} \psi_3 \right) \sqrt{a}\, d\xi$$

$$\Gamma_u = \left\{ (x_1, x_2)^T = \boldsymbol{\phi}(\xi) + u_1(\xi)\mathbf{a^1}(\xi) + u_3(\xi)\mathbf{a^3}(\xi), \xi \in [0, L] \right\}$$

Strong form of the fluid equations

Find the velocity $\mathbf{v} : \Omega_u^F \to \mathbb{R}^2$ and the pressure $p : \Omega_u^F \to \mathbb{R}$ such that:

$$
\begin{aligned}
-\mu \Delta \mathbf{v} + \nabla p &= \mathbf{f}^F, \quad \text{in } \Omega_u^F \\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_u^F \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_1 \\
\mathbf{n} \cdot \left(\sigma^F \mathbf{n}\right) &= -P_{in}, \quad \text{on } \Sigma_1 \\
\mathbf{v} &= 0, \quad \text{on } \Sigma_2 \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_3 \\
\mathbf{n} \cdot \left(\sigma^F \mathbf{n}\right) &= -P_{out}, \quad \text{on } \Sigma_3 \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Gamma_u \\
\mathbf{n} \cdot \left(\sigma^F \mathbf{n}\right) &= -\lambda^3 \circ \left(\phi + u_1 \mathbf{a}^1 + u_3 \mathbf{a}^3\right)^{-1}, \quad \text{on } \Gamma_u
\end{aligned}
$$

Week formulation of the fluid problem. Notations

$$\begin{aligned}
\mathbf{W} &= \left\{ \mathbf{w} \in \left(H^1\left(\Omega_u^F\right)\right)^2;\ \mathbf{w} = 0 \text{ on } \Sigma_2,\ w_2 = 0 \text{ on } \Sigma_1 \cup \Sigma_3 \right\}, \\
Q &= L^2\left(\Omega_u^F\right), \quad Z = H^{1/2}(\Gamma_u).
\end{aligned}$$

$$\begin{aligned}
a_F(\mathbf{v},\mathbf{w}) &= 2\mu \sum_{i,j=1}^{2} \int_{\Omega_u^F} \epsilon_{ij}(\mathbf{v})\epsilon_{ij}(\mathbf{w})\, d\mathbf{x} \\
b_F(\mathbf{w},q) &= -\int_{\Omega_u^F} (\nabla\cdot\mathbf{w})\, q\, d\mathbf{x} \\
c_F(\mathbf{w},\zeta) &= -\int_{\Gamma_u} (\mathbf{w}\cdot\boldsymbol{\tau})\, \zeta\, d\gamma \\
\ell_F(\mathbf{w}) &= \int_{\Omega_u^F} \mathbf{f}^F\cdot\mathbf{w}\, d\mathbf{x} - \int_{\Gamma_u} (\mathbf{w}\cdot\mathbf{n})\, \lambda^3 \circ \left(\phi + u_1\mathbf{a^1} + u_3\mathbf{a^3}\right)^{-1} d\gamma \\
&\quad - \int_{\Sigma_1} (\mathbf{w}\cdot\mathbf{n})\, P_{in}\, d\gamma - \int_{\Sigma_3} (\mathbf{w}\cdot\mathbf{n})\, P_{out}\, d\gamma
\end{aligned}$$

Week formulation of the fluid problem

Find $\mathbf{v} \in \mathbf{W}$, $p \in Q$ and $\eta \in Z$ such that

$$\begin{aligned} a_F(\mathbf{v}, \mathbf{w}) + b_F(\mathbf{w}, p) + c_F(\mathbf{w}, \eta) &= \ell_F(\mathbf{w}), \ \forall \mathbf{w} \in \mathbf{W} \\ b_F(\mathbf{v}, q) &= 0, \ \forall q \in Q \\ c_F(\mathbf{v}, \zeta) &= 0, \ \forall \zeta \in Z \end{aligned}$$

The physical meaning of η is that of the tangential stresses on the interface, i.e. $\eta = \boldsymbol{\tau} \cdot \sigma^F \mathbf{n}$.

Continuity of the stresses and of the velocity on the interface

The surface forces acting to the arch in a point $\phi(\xi)$ of Γ_0 are given by

$$\lambda^1 \mathbf{a_1} + \lambda^3 \mathbf{a_3},$$

while the surface forces acting to the fluid in the point $\phi(\xi) + u_1(\xi)\mathbf{a^1}(\xi) + u_3(\xi)\mathbf{a^3}(\xi)$ of Γ_u are given by

$$\sigma^F \mathbf{n} = \left(\boldsymbol{\tau} \cdot \left(\sigma^F \mathbf{n}\right)\right) \boldsymbol{\tau} + \left(\mathbf{n} \cdot \left(\sigma^F \mathbf{n}\right)\right) \mathbf{n}.$$

$$\frac{1}{\sqrt{a}} \mathbf{a_1} = \mathbf{a^1} \approx \boldsymbol{\tau}, \quad \mathbf{a_3} = \mathbf{a^3} \approx \mathbf{n}$$

$$-\lambda^1 \sqrt{a} \approx \boldsymbol{\tau} \cdot \left(\sigma^F \mathbf{n}\right) = \eta, \quad -\lambda^3 \approx \mathbf{n} \cdot \left(\sigma^F \mathbf{n}\right)$$

$$\mathbf{v} \cdot \mathbf{n} = 0, \quad \mathbf{v} \times \mathbf{n} = 0, \text{ on } \Gamma_u$$

Coupling fluid and arch equations

For a given $\boldsymbol{\lambda}$, we obtain the displacement $\mathbf{u}$ by solving the arch problem.
From the fluid problem, we get the velocity $\mathbf{v}$, the pressure p of the fluid and the tangential stresses on the interface η.
The fluid-arch interaction problem is to find $\boldsymbol{\lambda}$ such that

$$\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= 0, \text{ on } \Gamma_u \\ -\lambda^1 \sqrt{a} &= \eta \circ \left(\phi + u_1 \mathbf{a^1} + u_3 \mathbf{a^3} \right), \text{ on } [0, L]. \end{aligned}$$

Optimal control problem

We assume in the following that the displacement of the arch depends only on λ^3.

$$\inf \frac{1}{2} \int_{\Gamma_u} (\mathbf{v} \cdot \mathbf{n})^2 \ d\gamma$$

subject to

$$\lambda^3 \in L^2\left(]0, L[\right)$$

$$u \text{ solution of the arch problem}$$

$$\mathbf{v}, \ p, \ \eta \text{ solution of the fluid problem}$$

Geometry

Let $L = 6\ cm$.

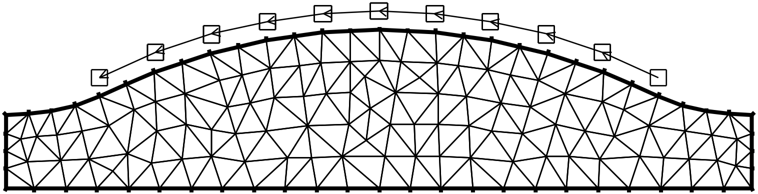
$$\Gamma_0 = \left\{ (x_1, x_2) \in \mathbb{R}^2;\ x_1 = \xi,\ x_2 = -5 + \sqrt{45 - (\xi - 3)^2},\ \xi \in [0, L] \right\}.$$

Physical parameters

The inflow Σ_1 and outflow Σ_3 sections are segments of length 0.8.
The thickness of wall $e = 0.1\ cm$, the Young's modulus $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the Poisson's ratio $\nu = 0.49$ and the mass density $\rho^S = 1.1\ \frac{g}{cm^3}$.
The viscosity of the fluid $\mu = 0.035\ \frac{g}{cm \cdot s}$, the volume force in fluid is $\mathbf{f}^F = (0, 0)^T$ and the outflow pressure $P_{out} = 0$.

Finite Element approximation

The normal displacement of the arch is approached by the finite element $\mathbb{P}_3$ Hermite, while the tangential displacement is approached by $\mathbb{P}_1$ on segments.
For the approximation of the fluid velocity and pressure we have employed the triangular finite elements $\mathbb{P}_1$+*bubble* and $\mathbb{P}_1$ respectively. The tangential stresses on the interface is approached by the finite element $\mathbb{P}_1$ on segments.



Construction of fluid meshes

All the fluid meshes are obtained by moving a fixed mesh with a displacement which is the solution of a Laplace problem with Dirichlet boundary conditions.
On the fixed boundaries Σ_1, Σ_2, Σ_3 the mesh displacement vanishes, while on the interface Γ_u, it is equal to the arch displacement.

Solving the optimal control problem

In order to solve numerically the optimal control problem, we have used the Broyden, Fletcher, Goldforb, Shano (BFGS) method where the gradient of the cost function was approached by the first order Finite Difference Method with the grid spacing 0.01.
We can use the stopping criteria $|J| < \epsilon_1$ or $\|\nabla J\|_\infty < \epsilon_2$.

Fluid mesh with 20 segments on the interface

Iteration	J
0	2547.091786
1	1211.782266
2	725.6283591
3	510.1113951
4	55.08303685
5	14.69322181
6	2.761055109
7	1.355973240

Iteration	J
8	1.023813629
9	0.541353106
10	0.196142907
11	0.074620024
12	0.061075290
13	0.059283206
14	0.059257208

$$\|\nabla J\|_\infty = 1.9 \cdot 10^2$$

Fluid mesh with 50 segments on the interface

Iteration	J
0	2961.338390
1	2160.756802
2	1392.917188
3	879.6830449
4	206.9249346
5	28.11455274
6	18.81352198

Iteration	J
7	8.851579284
8	2.369258721
9	1.271752295
10	0.382701032
11	0.179680309
12	0.124469558
13	0.123437472

$\|\nabla J\|_{\infty} = 1.0 \cdot 10^2$

Fluid mesh with 80 segments on the interface

Iteration	J	Iteration	J
0	3619.097215	6	29.86861359
1	1503.175625	7	13.36326171
2	258.7993032	8	10.33965940
3	91.91149097	9	0.753294422
4	61.75894998	10	0.095747979
5	42.29953564	11	0.092956444

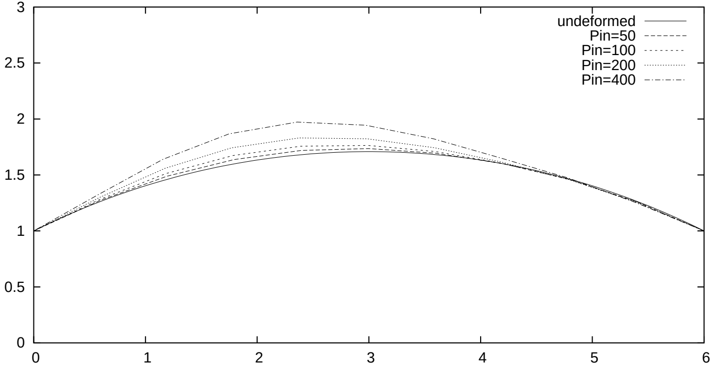
$\|\nabla J\|_\infty = 3.3 \cdot 10^2$

Results obtained by BFGS method for different inflow pressure

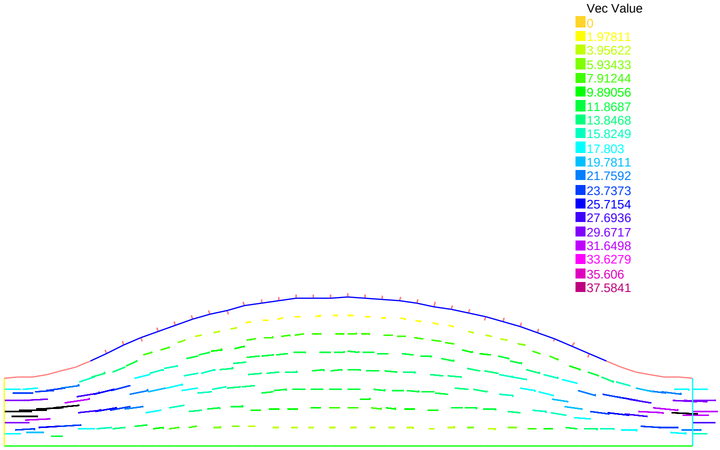
The arch mesh has 10 segments. All the fluid meshes have 516 triangles, 304 vertices and 30 segments on the elastic boundary Γ_u. We have solved numerically the fluid-structure problem for following inflow pressure $P_{in} = 50,\ 100,\ 200,\ 400\ dyn/cm^2$. The stopping criteria was: $|J| < 0.5$ or $\|\nabla J\|_\infty < 0.1$.

P_{in}	initial J	final J	no. BFGS iterations
50	2928.63	0.23	9
100	356.86	0.26	13
200	1378.34	0.56	17
400	5257.25	2.08	20

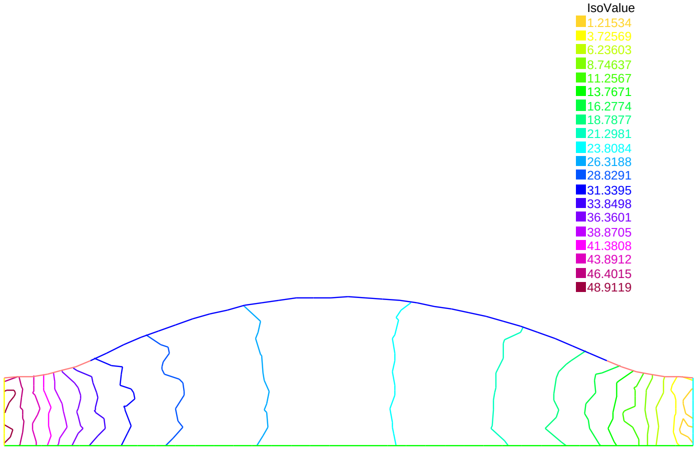
Arch deformations for different inflow pressures



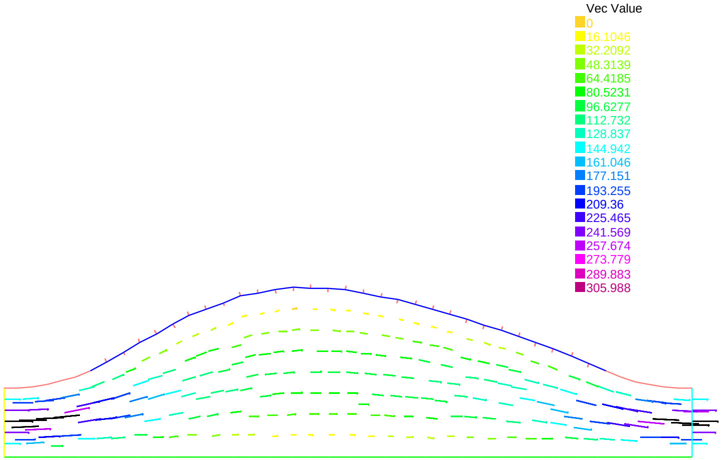
Fluid velocities scaled by a factor 0.008 in the case $P_{in} = 50$



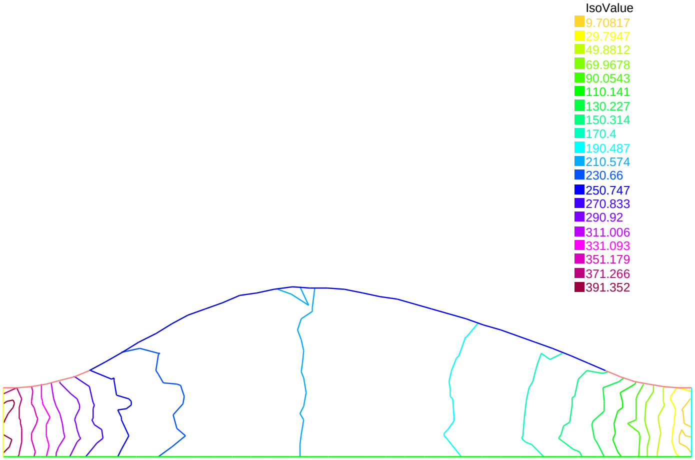
Fluid pressure in the case $P_{in} = 50$



Fluid velocities scaled by a factor 0.001 in the case $P_{in} = 400$



Fluid pressure in the case $P_{in} = 400$



Conclusions

We have formulated a fluid-structure interaction as an optimal control problem. The control is the normal force acting on the interface and the observation is the normal velocity of the fluid on the interface.
The BFGS method finds numerically small residual function even for a reduced number of controls.