Optimal control approach for the fluid-structure interaction problems

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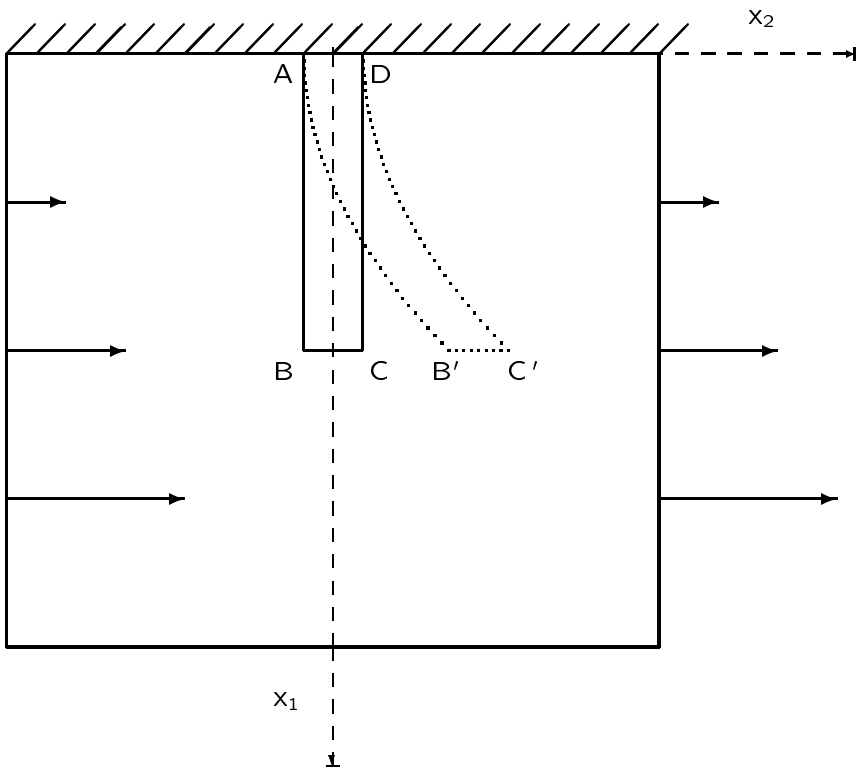


The deformation of the beam is described using the displacement of the median thread

$$u = (u_1, u_2) : [0, L_1] \to \mathbb{R}^2.$$

For instant, we assume that $u_1 = 0$.

The domain occupied by the fluid is Ω_u^F.

The contact surface between fluid and beam is

$$\Gamma_u = \Big]AB'\Big[\cup \Big[B'C'\Big] \cup \Big]C'D\Big[.$$

The other boundary of the fluid domain is noted

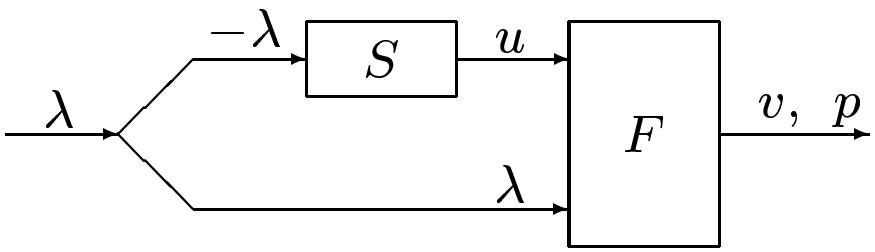
$$\Gamma_1$$

Fluid structure interaction

In the steady case, the fluid and structure equations are coupled via two boundary conditions

$$\begin{aligned} v &= 0 \quad \text{on } \Gamma_u, \\ \sigma^F n &= \sigma^S n \quad \text{on } \Gamma_u. \end{aligned}$$

We denote by $\lambda = \sigma^F n$ the forces induced by the beam on the contact surface. Consequently, $-\lambda$ represent the forces induced by the fluid acting to the beam.



We search to find out λ, such that $v_{|\Gamma_u} = 0$. This is a exact controllability problem.

Unconventional boundary conditions

The stress tensor for a Newtonian incompressible fluid has the form

$$\sigma_{ij}^F = \mu \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right) - p\delta_{ij}.$$

We have to solve the fluid equations with the boundary condition

$$\sigma^F n = \lambda, \text{ on } \Gamma_u,$$

but it isn't appropriate: the fluid can cross the beam.

For the fluid, we shall use on the contact boundary

$$\begin{aligned} v \cdot n &= 0, \\ p &= p_0. \end{aligned}$$

We suppose that the forces induced by the fluid acting to the beam have the form

$$-\lambda = p_0 n.$$

Beam equations. Variational formulation

$$U = \left\{ \phi \in H^2\left(]0, L_1[\right);\ \phi(0) = \phi'(0) = 0 \right\}$$

$$\begin{cases} a_S : U \times U \to \mathbb{R} \\ a_S(\phi, \psi) = D_2 \displaystyle\int_{]0, L_1[} \frac{d^2\phi}{dx_1^2}(x_1)\, \frac{d^2\psi}{dx_1^2}(x_1)\, dx_1 \end{cases}$$

where $D_2 > 0$ is a given constant.

Proposition Let $f_2^S \in L^2\left(]0, L_1[\right)$ and $\eta_2 \in L^2\left(]0, L_1[\right)$. Then the problem: find u_2 in U such that

$$\begin{aligned} a_S(u_2, \psi) \;=\; & \int_{]0, L_1[} \eta_2(x_1)\, \psi(x_1)\, dx_1 \\ & + \int_{]0, L_1[} f_2^S(x_1)\, \psi(x_1)\, dx_1, \quad \forall \psi \in U \end{aligned}$$

has a unique solution.

We denote by f_2^S the external volume forces and by η_2 the external surface forces (the forces exerted by the fluid).

Beam equations. Strong formulation

$$u_2''''(x_1) = \frac{1}{D_2}\left(\eta_2(x_1) + f_2^S(x_1)\right), \quad \forall x_1 \in]0, L_1[$$
$$u_2(0) = u_2'(0) = 0,$$
$$u_2''(L_1) = u_2'''(L_1) = 0.$$

For $f_2^S = 0$ and $\eta_2(x_1) = \alpha + \beta x_1 + \gamma x_1^2$, we obtain

$$\begin{aligned} u_2(x_1) \;=\; & \frac{1}{360 D_2}\Big[15\left(x_1^2 - 4x_1 L_1 + 6L_1^2\right)\alpha \\ & +3\left(x_1^3 - 10x_1 L_1^2 + 20L_1^3\right)\beta \\ & +\left(x_1^4 - 20x_1 L_1^3 + 45L_1^4\right)\gamma\Big] \end{aligned}$$

Fluid equations. Strong formulation

Find the velocity $v : \overline{\Omega}_u^F \to \mathbb{R}^2$, the pressure $p : \overline{\Omega}_u^F \to \mathbb{R}$ and the vorticity $\omega : \overline{\Omega}_u^F \to \mathbb{R}$, such that

$$
\begin{cases}
\dfrac{\partial p}{\partial x_1} + \mu \dfrac{\partial \omega}{\partial x_2} &= f_1^F \quad \text{in } \Omega_u^F \\
\dfrac{\partial p}{\partial x_2} - \mu \dfrac{\partial \omega}{\partial x_1} &= f_2^F \quad \text{in } \Omega_u^F \\
\omega - \dfrac{\partial v_2}{\partial x_1} + \dfrac{\partial v_1}{\partial x_2} &= 0 \quad \text{in } \Omega_u^F \\
\operatorname{div} v &= 0 \quad \text{in } \Omega_u^F \\
v &= g \quad \text{on } \Gamma_1 \\
v \cdot n &= 0 \quad \text{on } \Gamma_u \\
p &= p_0 \quad \text{on } \Gamma_u \\
\omega &= 0 \quad \text{on } \Gamma_1.
\end{cases}
$$

This is the two dimensional Stokes equations in the velocity-pressure-vorticity formulation.

Here, $\mu > 0$ is the dynamic viscosity of the fluid, f^F is the external given force per unit volume.

Fluid equations. Variational formulation

$$
\begin{aligned}
W &= \left\{ w \in H^1\left(\Omega_u^F\right)^2 ; w = 0 \text{ on } \Gamma_1,\ w \cdot n = 0 \text{ on } \Gamma_u \right\} \\
Q &= \left\{ q \in H^1\left(\Omega_u^F\right) ;\ q = 0 \text{ on } \Gamma_u \right\} \\
M &= \left\{ \omega \in H^1\left(\Omega_u^F\right) ;\ \omega = 0 \text{ on } \Gamma_1 \right\}
\end{aligned}
$$

Proposition For all u_2 in U, and f^F in $L^2\left(\Omega_u^F\right)^2$, the problem:
Find $v - v_0 \in W$, $p - p_0 \in Q$, $\omega \in M$, such that

$$
\begin{aligned}
&\left(\frac{\partial p}{\partial x_1} + \mu \frac{\partial \omega}{\partial x_2}, \frac{\partial q}{\partial x_1} + \mu \frac{\partial \rho}{\partial x_2} \right) \\
&+ \left(\frac{\partial p}{\partial x_2} - \mu \frac{\partial \omega}{\partial x_1}, \frac{\partial q}{\partial x_2} - \mu \frac{\partial \rho}{\partial x_1} \right) \\
&+ \left(\omega - \frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}, \rho - \frac{\partial w_2}{\partial x_1} + \frac{\partial w_1}{\partial x_2} \right) \\
&+ (\operatorname{div} v, \operatorname{div} w) \\
&= \left(f_1^F, \frac{\partial q}{\partial x_1} + \mu \frac{\partial \rho}{\partial x_2} \right) \\
&+ \left(f_2^F, \frac{\partial q}{\partial x_2} - \mu \frac{\partial \rho}{\partial x_1} \right), \quad \forall w \in W, \forall q \in Q, \forall \rho \in M
\end{aligned}
$$

has a unique solution.

Optimal control approach

$$\inf J(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) = \frac{1}{2} \|v \cdot \tau\|^2_{0,\Gamma_u}$$

subject to:

$$(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2) \in K \subset \mathbb{R}^6$$

$$\begin{aligned} u_2(x_1) \;=\; & \frac{1}{360 D_2} \Big[15 \left(x_1^2 - 4 x_1 L_1 + 6 L_1^2 \right) (\alpha_1 - \alpha_2) \\ & + 3 \left(x_1^3 - 10 x_1 L_1^2 + 20 L_1^3 \right) (\beta_1 - \beta_2) \\ & + \left(x_1^4 - 20 x_1 L_1^3 + 45 L_1^4 \right) (\gamma_1 - \gamma_2) \Big] \end{aligned}$$

(v, p, ω) solution of the Stokes equations with

$$p_0(x_1, x_2) = \begin{cases} \left(\alpha_1 + \beta_1 x_1 + \gamma_1 x_1^2 \right), \text{ on }]A, B'] \\ \left(\alpha_2 + \beta_2 x_1 + \gamma_2 x_1^2 \right), \text{ on }]D, C'] \\ E\left(p_0(B'), p_0(C') \right), \text{ on }]B', C'[\end{cases}$$

It's an optimal control problem with:

- Dirichlet boundary control ($p_{0|\Gamma_u}$),
- Dirichlet boundary observation ($v_{|\Gamma_u}$).

Remarks

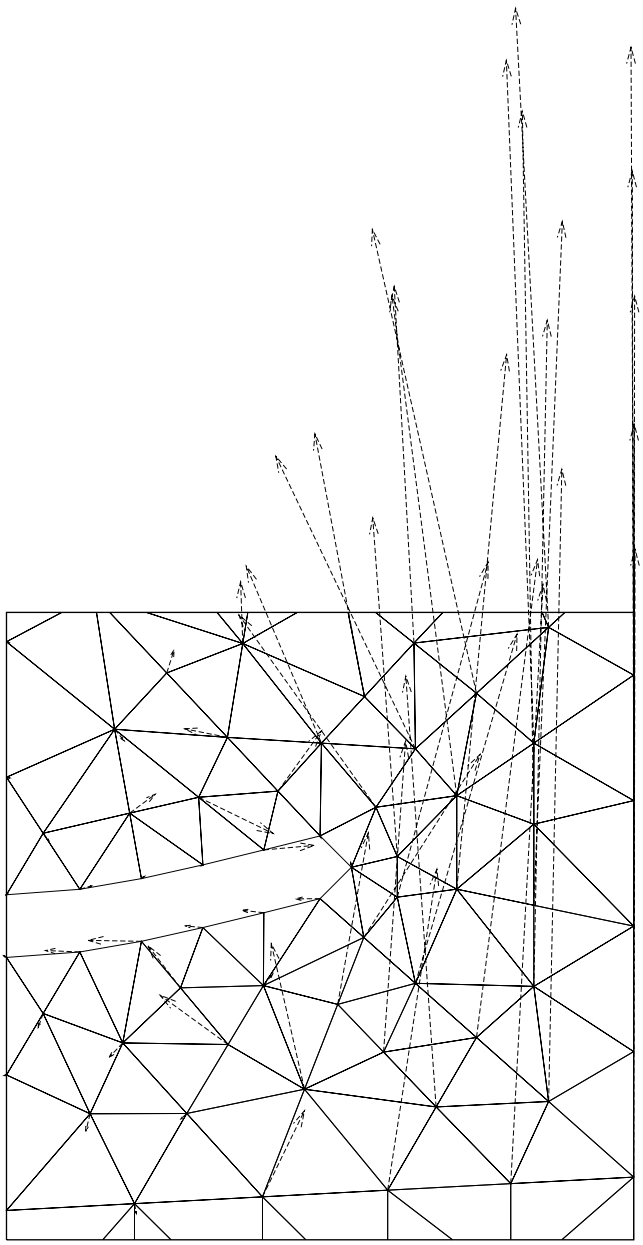
The existence of an optimal control could be find in [C.M. Murea, Y. Maday, 1996] for a related problem. In [C.M. Murea, C. Vazquez] it is proved the differentiability of the cost function and it is given the analytic formula for the gradient.

An open problem is to find additional conditions in order to obtain zero for the optimal value of the cost function. This is an approximate controllability problem. For a linear model (the domain of the fluid doesn't depend upon the displacement of the structure), we can find approximate controllability results in [J. L. Lions, E. Zuazua, *ESAIM: Contr. Optim. Calc. Var.* 1995], [A. Osses, J. P. Puel, *ESAIM: Contr. Optim. Calc. Var.* 1999] and [A. Osses, *SIAM J. Control Optim.*, to appear].

If $v \cdot \tau$ is constant on Γ_u, then v is constant on Γ_u. We can prove that $\left(\sigma^F n\right) \cdot n = -p_0$. Consequently, solving the beam equations under the action of the surface forces $-\lambda = p_0\, n$ on Γ_u is reasonable.

Numerical tests

Its have been produced using *freefem+*.

The boundary condition $v \cdot n = 0$ on Γ_u was replaced by $v_2 = 0$.

The computed velocity isn't a divergence-free field. For a better approximation of the incompressibility condition, we can penalize the term $(\operatorname{div} v, \operatorname{div} w)$ in the variational formulation.