SENSITIVITY AND APPROXIMATION OF COUPLED FLUID-STRUCTURE EQUATIONS BY VIRTUAL CONTROL METHOD

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Abstract

The formulation of a particular fluid-structure interaction as an optimal control problem is the departure point of this work. The control is the vertical component of the force acting on the interface and the observation is the vertical component of the velocity of the fluid on the interface. This approach permits to solve the coupled fluid-structure problem by partitioned procedures. The analytic expression for the gradient of the cost function is obtained in order to devise accurate numerical methods for the minimization problem. Numerical results arising from blood flow in arteries are presented. To solve numerically the optimal control problem, we use a quasi Newton method which employs the analytic gradient of the cost function and the approximation of the inverse Hessian is updated by the Broyden, Fletcher, Goldforb, Shano (BFGS) scheme. This algorithm is faster than fixed point with relaxation or block Newton methods.

Key Words. fluid-structure interaction; virtual control; mixed formulations; optimization algorithms; sensitivity analysis.

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1 Introduction

In this paper we consider a variable bounded domain which is occupied by a steady newtonian incompressible creeping fluid. The boundary can be decomposed into a rigid part and an elastic part.

The mathematical model which governs the fluid is based on a steady Stokes equation while the deformation of the elastic part of the boundary verifies a particular beam equation without shearing stress. Therefore the solution of the model consists of the determination of the elastic boundary displacement and the computation of the velocity and the pressure in the fluid domain.

In a first sense, the physical problem is related with those treated in fluid-structure interaction literature but the vibration approach is not considered here.[30] In other sense, the asymptotic limit when the fluid domain width tends to zero can be modeled by a one-dimensional approach of Stokes equation, i.e. Reynolds equation, widely used in lubrication theory.[3]

On the other hand, if we think about the elastic boundary as part of the boundary of a two-dimensional domain which is unknown a priori, then the problem can be framed as a free boundary like problem. The free boundary aspect of the model motivates the need of two coupling boundary conditions: continuity of the velocity and of the stresses across the interface fluid-structure.

This kind of problem is of considerable interest in biomechanics (the simulation of blood flow in large arteries, [29], [17], [33], [8], [18], [38]), in aeroelasticity (fluttering of wings, [13], [14], [35], [36]), in cars industry (design of hydraulic shock absorber, [26]).

The existence results for the fluid-structure interaction can be found in [21], [23], [2] for the steady case and in [22], [12], [4] for the unsteady case.

Sensitivity analysis of a coupled fluid-structure system was investigated in [15].

The most frequently, the fluid-structure interaction problems are solved numerically by partitioned procedures, i.e. the fluid and the structure equations are solved separately, which allows to use the existing solvers for each subproblem.

There are different strategies to discretise in time the unsteady fluid-structure interaction problem. A family of explicit algorithms known also as staggered was successfully employed for the aeroelastic applications.[13] Their stability properties were studied in [35] and [36]. For the stability reason, a very small time step is necessary.

As it shown in [26] and [33], the staggered algorithms are unstable when the structure is light and its density is comparable to that of its fluid. In order to obtain unconditionally stable algorithms, at each time step we have to solve a non-linear fluid-structure coupled system. This can be done using fixed point strategies with eventually a relaxation parameter, but it has slow convergence rate [26], [33], [17]. The convergence can be accelerated using Aitken's method [18] or transpiration condition [11].

Other way to accelerate the convergence is to use methods which employ the derivative. In [40] a block Newton algorithm was used where the derivative of the operators are approached by finite differences. Good convergence rate was obtained in [18] where the derivative of the operator was replaced by a much simpler operator. The block Schur-Newton method is proposed in [16] where the derivatives of the fluid and structure operators with respect to the state variables were computed exactly, but this algorithm has not been implemented yet.

In a previous work, a three-dimensional fluid-structure interaction was formulated as an optimal control system, where the control is the force acting on the interface and the observation is the velocity of the fluid on the interface.[32] The fluid equations were solved taking into account a given surface force on the interface. The existence of an optimal control was proved. We have to precise that the fluid-structure interaction problem and its optimal control version are not equivalent.

In this work, a two-dimensional steady state fluid-structure coupled problem is approximated by an optimal control system, where the control is the vertical component of the force acting on the interface and the observation is the vertical component of the velocity of the fluid on the interface. The control approach permits to solve the coupled fluid-structure problem by partitioned procedures.

The analytic computation of the gradient for the cost function is one of the main goals of this work in order to apply accurate numerical methods. Moreover, from the theoretical viewpoint, the optimality conditions can be written in terms of this analytic expression of the gradient. In fact, although the analytic formula for the gradient involves the solution of several auxiliary problems, the alternative use of finite difference approximations for the derivatives introduces truncation errors and it is potentially much more sensitive to ill-conditioning of the state equations.[27]

The aims of this paper are: to analyse the behavior of the fluid and structure sub-problems under the variation of the force acting on the interface, to prove the differentiability of the cost function and to present numerical results arising from blood flow in arteries. To solve numerically the optimal control problem, we use a quasi Newton method which employs the analytic gradient of the cost function and the approximation of the inverse Hessian is updated by the Broyden, Fletcher, Goldforb, Shano (BFGS) scheme. This algorithm is faster than fixed point with relaxation or block Newton methods.

In Section 2 the particular fluid-structure problem is presented, related notations are introduced and the associated optimal control problem is briefly posed. In Section 3 the weak formulation of the structure equations is analysed and we precise the set of admissible controls. For a given structure displacement, the mixed formulations governing the fluid velocity and pressure are posed in the eulerian and arbitrary lagrangian eulerian coordinates in Sections 4 and 5, respectively. In these arbitrary lagrangian eulerian coordinates the optimal control system is detailed in Section 6. Next, the continuity and the differentiability of the cost function are proved in the Section 7 and 8. Moreover, the exact expression of the cost function gradient is obtained. In Section 9 we present an interesting application to blood flow simulation in medium vessels. For this, particular methods to solve the structure and fluid equations as well as specific algorithms for the discrete optimization problem are proposed. Some numerical results for real data are presented and discussed. The last section is devoted to some concluding remarks.

2 Presentation of the problem

In order to pose the equations for the model let us introduce some mathematical notations. Let L and H be two positive constants. We introduce the classical Sobolev space $U = H_0^2(0,L)$ and the sets (see the Figure 1):

$$\Omega_0^F = (0,L)\times(0,H), \quad \Gamma_0 = (0,L)\times\{H\}, \quad \Sigma_1 = \{0\}\times(0,H),$$
$$\Sigma_2 = (0,L)\times\{0\}, \quad \Sigma_3 = \{L\}\times(0,H), \quad \Sigma = \Sigma_1\cup\overline{\Sigma_2}\cup\Sigma_3.$$

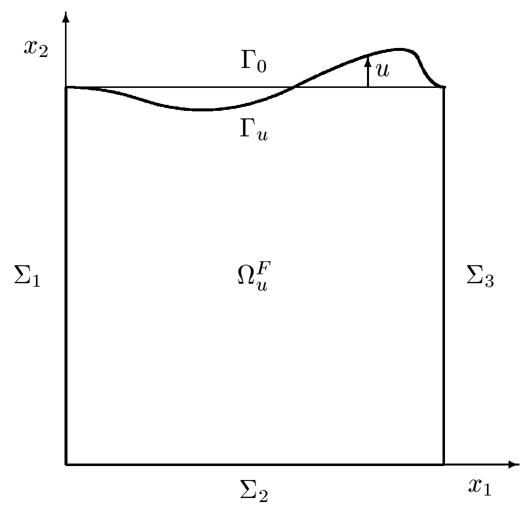

Figure 1: Sets appearing in the fluid-structure problem

As we have the continuous and compact inclusion of $H_0^2(0,L)\subset\mathcal{C}^1(0,L)$ then for each $u\in U$ we denote by u' its derivative (in fact it is a classical derivative) and by u'' its second (weak) derivative. For a given $e\in(0,H)$ we define the set

$$\begin{aligned}\mathcal{U}_{ad} \quad = \quad & \{u\in U;\, u(0) = u(L) = u'(0) = u'(L) = 0,\\ & \left.\int_0^L u(x_1)\,dx_1 = 0,\ H + u(x_1)\geq e, \forall x_1\in[0,L]\right\}.\end{aligned}$$

Moreover, for each $u\in\mathcal{U}_{ad}$, we introduce the notations (see the Figure 1)

$$\begin{aligned}\Omega_u^F &= \left\{(x_1,x_2)\in\mathbb{R}^2;\, x_1\in(0,L),\, 0<x_2<H+u(x_1)\right\},\\ \Gamma_u &= \left\{(x_1,x_2)\in\mathbb{R}^2;\, x_1\in(0,L),\, x_2 = H+u(x_1)\right\}.\end{aligned}$$

In view of the definition of the $\mathcal{U}_{ad}$, the two-dimensional domain occupied by the fluid is Ω_u^F, the elastic interface between fluid and structure is the free boundary Γ_u, while Σ represents the rigid boundary.

We suppose that the fluid is governed by the steady Stokes equations, while the deformation of the elastic part of the boundary verifies a particular beam equation without shearing stress.[5] We consider that the structure is a beam of axis parallel to Ox_1 with constant thickness h. We assume that the displacement of the beam is normal to its axis.

The problem is to find:

- $u : [0, L] \to \mathbb{R}$ the displacement of the structure,
- $v = (v_1, v_2)^T : \Omega_u^F \to \mathbb{R}^2$ the velocity of the fluid and
- $p : \Omega_u^F \to \mathbb{R}$ the pressure of the fluid,

such that

$$\begin{aligned}
EI\, u''''(x_1) &= -\left(\sigma^F n \cdot e_2\right)_{(x_1, H+u(x_1))} \sqrt{1 + (u'(x_1))^2} + f^S(x_1) && (1)\\
u(0) &= u(L) = u'(0) = u'(L) = 0 && (2)\\
\int_0^L u(x_1)\, dx_1 &= 0 && (3)\\
e &\leq \inf_{x_1 \in [0,L]} \{H + u(x_1)\} && (4)\\
-\mu \Delta v + \nabla p &= f^F, \quad in\ \Omega_u^F && (5)\\
\text{div } v &= 0, \quad in\ \Omega_u^F && (6)\\
v &= g, \quad on\ \Sigma && (7)\\
v &= 0, \quad on\ \Gamma_u && (8)
\end{aligned}$$

where

- $EI = \frac{Eh^3}{12}$ is rigidity to bending modulus of the structure, E is the Young modulus, h is the thickness.
- $f^S : (0, L) \to \mathbb{R}$ are the averaged volume forces of the structure, in general the gravity forces and in this case we have $f^S(x_1) = -g_0 \rho^S h$, where g_0 is the gravity, ρ^S is the density of the structure,
- $\mu > 0$ is the viscosity of the fluid,
- $f^F = (f_1^F, f_2^F)^T : \Omega_u^F \to \mathbb{R}^2$ are the volume forces of the fluid, in general the gravity forces,
- $g = (g_1, g_2)^T : \Sigma \to \mathbb{R}^2$ is the imposed velocity profile of the fluid on the rigid boundary, such that

$$\int_\Sigma g \cdot n\, d\sigma = 0 \qquad (9)$$

- $\sigma^F = -p\, I + \mu \left(\nabla v + \nabla v^T\right)$ is the stress tensor of the fluid,

- $n = (n_1, n_2)^T$ the unit outward normal vector to $\partial\Omega_u^F$,
- $e_2 = (0,1)^T$ is the unit vector in the x_2 direction.

The incompressibility of the fluid (6) states that the volume of the fluid is conserved or equivalently $\int_0^L u(x_1)\,dx_1$ is constant. Without loss of generality, we assume that this constant is zero and we obtain the condition (3).

The inequality (4) implies that the fluid domain is connected. The constant e has not a physical meaning.

The system (1)-(8) is a coupled fluid-structure problem.

The displacement of the structure depends on the vertical component of the stresses exerced by the fluid on the interface (equation 1). This cames from the continuity of the stresses across the interface.

The movement of the structure changes the domain where the fluid equations must be solved (equations 5,6). Also, on the interface we have to impose the equality between the fluid and structure velocity (equation 8).

We shall introduce the control approach.

Let $\widehat{\lambda} : (0, L) \to \mathbb{R}$ be the control function.

The displacement of the structure is computed by

$$EI\,u''''(x_1) = -\widehat{\lambda}(x_1) + f^S(x_1), \quad \forall x_1 \in (0, L)$$

with boundary conditions (2), such that (3) and (4) hold.

We can compute the velocity and the pressure of the fluid as the solution of the equations (5), (6) with boundary conditions on the rigid boundary (7) together with boundary conditions on the interface: $v_1 = 0$ and

$$\left(\sigma^F n \cdot e_2\right)_{(x_1, H+u(x_1))} = \frac{\widehat{\lambda}(x_1)}{\sqrt{1 + \left(u'(x_1)\right)^2}}, \quad \forall x_1 \in (0, L).$$

The control problem is to find $\widehat{\lambda}$, such that $v_2 = 0$ on Γ_u.

As we use the value $-\widehat{\lambda}$ for the applied stresses in the equations of the structure and we take the value $\widehat{\lambda}$ in the equations of the fluid, the continuity of the stresses across the interface is strongly accomplished.

In the following, the boundary condition $v_{2|\Gamma_u} = 0$ is treated by the Least Square Method and we obtain the optimal control problem

$$\inf_{\widehat{\lambda}} \frac{1}{2} \left\| v_{2|\Gamma_u} \right\|^2 .$$

The control $\widehat{\lambda}$ and the cost function are "virtual". The idea of Virtual Control which leads to Domain Decomposition Methods was presented in [28] and in the references given there.

Next, we shall precise the regularity of the control which is linked to the equivalence or not-equivalence between the fluid-structure equations (1)–(8) and its optimal control version.

If the system of fluid-structure equations (1)–(8) has a strong solution $u \in H^4(0,L)$, $v \in \left(H^2\left(\Omega_u^F\right)\right)^2$ and $p \in H^1\left(\Omega_u^F\right)$, then the control given by the relation

$$\widehat{\lambda}(x_1) = \left(\sigma^F n \cdot e_2\right)_{(x_1, H+u(x_1))} \sqrt{1 + \left(u'(x_1)\right)^2}$$

belongs to $L^2(0,L)$. In fact, the control is even smoother. In this case, the system (1)–(8) is equivalent to the control problem. So, there exists $\widehat{\lambda} \in L^2(0,L)$ such that $v_{2|\Gamma_u} = 0$. In [4] the existence of a strong solution was proved for a related problem.

If the system of fluid-structure equations (1)–(8) has only a week solution $u \in H^2(0,L)$, $v \in \left(H^1\left(\Omega_u^F\right)\right)^2$ and $p \in L^2\left(\Omega_u^F\right)$, then $\widehat{\lambda}$ is well defined in a space like the dual of $H_{00}^{1/2}(0,L)$, which is larger than $L^2(0,L)$. In this case, the optimal control problem

$$\inf_{\widehat{\lambda} \in L^2(0,L)} \frac{1}{2} \left\| v_{2|\Gamma_u} \right\|^2$$

has not solution, so it is not equivalent to the fluid-structure equations (1)–(8). Using the density of $L^2(0,L)$ in the dual of $H_{00}^{1/2}(0,L)$, we could prove that $\inf \frac{1}{2} \left\| v_{2|\Gamma_u} \right\|^2 = 0$ for $\widehat{\lambda} \in L^2(0,L)$, but this aspect will not study here.

The existence of a weak solution was proved in [21] and [2] for a two-dimensional steady state fluid-structure interaction problem, in [23] for a three-dimensional steady state, in [22] and [12] for an unsteady state.

In the following, we shall take $\widehat{\lambda}$ in $L^2(0,L)$ because it is simpler to approximate than the dual of $H_{00}^{1/2}(0,L)$.

3 Weak formulation for the structure equations

In this paragraph we present the weak formulation for the structure equations. We have assumed that the structure is governed by a classical beam equations without shearing stress.[5]

So, for a given $EI \in \mathbb{R}_+^*$ which is the rigidity to bending modulus of the structure, we define the bilinear form

$$\left\{ \begin{array}{ccl} a_S : U \times U & \to & \mathbb{R} \\ (\phi, \psi) & \mapsto & a_S(\phi, \psi) = EI \displaystyle\int_0^L \phi''(x_1)\, \psi''(x_1)\, dx_1 \, . \end{array} \right. \tag{10}$$

The bilinear form a_S is evidently symmetric and continuous. In addition, applying the Poincaré inequality (see [10] vol. 3, chap. IV, p. 920), we obtain that a_S is U-elliptic. Moreover, let U' be the dual of U. We denote by $\langle \cdot, \cdot \rangle_{U',U}$ the duality pairing between U' and U. A simple consequence of the Lax-Milgram Theorem (see [10] vol. 4, chap. VII, p. 1217) leads to the following result:

Proposition 1 *Let $f^S \in U'$ and $\eta \in L^2(0,L)$. Then, the problem: Find $u \in U$ such that*

$$a_S(u,\psi) = \int_0^L \eta(x_1)\,\psi(x_1)\,dx_1 + \left\langle f^S, \psi\right\rangle_{U',U} \quad \forall \psi \in U \tag{11}$$

has a unique solution. Moreover the solution $u \in \mathcal{C}^1([0,L])$ and we have the $L^\infty(0,L)$ estimate:

$$\|u\|_{L^\infty(0,L)} \le C_1 \|\eta\|_{L^2(0,L)} + C_2 \left\|f^S\right\|_{U'}$$

where C_1 and C_2 are constants.

When the data and the solution are smooth enough the solution u verifies the strong formulation given by:

$$\begin{aligned} EI\,u''''(x_1) &= \eta(x_1) + f^S(x_1),\ \forall x_1 \in (0,L) \\ u(0) &= u'(0) = 0, \\ u(L) &= u'(L) = 0. \end{aligned}$$

Remark 1 *The physical meaning of f^S is that of an external force applied to the elastic structure. For example, the consideration of an harmonic expression for f^S would lead to an harmonic response of the fluid-structure device. Also, the gravity forces are included in f^S. In the coupled model, η is associated to the fluid forces acting on the structure.*

In order to obtain a fluid domain with constant volume, we have to impose some condition to η. We denote by $L_0^2(0,L) = \left\{\eta \in L^2(0,L); \int_0^L \eta(x_1)\,dx_1 = 0\right\}$.

Proposition 2 *Let $f^S \in U'$ and $\eta \in L_0^2(0,L)$.*

i) Then there exist an unique $u \in U$, such that $\int_0^L u(x_1)\,dx_1 = 0$ and an unique constant $c \in \mathbb{R}$ solutions of

$$a_S(u,\psi) = \int_0^L (\eta(x_1) + c)\,\psi(x_1)\,dx_1 + \left\langle f^S, \psi\right\rangle_{U',U} \quad \forall \psi \in U \tag{12}$$

ii) Let $u_0 \in U$, such that $\int_0^L u_0\,dx_1 = 0$ and $c_0 \in \mathbb{R}$ are the solution of

$$a_S(u_0,\psi) = c_0 \int_0^L \psi(x_1)\,dx_1 + \left\langle f^S, \psi\right\rangle_{U',U} \quad \forall \psi \in U \tag{13}$$

and $u_\eta \in U$, such that $\int_0^L u_\eta\,dx_1 = 0$ and $\ell(\eta) \in \mathbb{R}$ are the solution of

$$a_S(u_\eta,\psi) = \int_0^L (\eta(x_1) + \ell(\eta))\,\psi(x_1)\,dx_1 \quad \forall \psi \in U. \tag{14}$$

Then, $u = u_0 + u_\eta$, $c = c_0 + \ell(\eta)$ and the applications

$$\eta \in L_0^2(0,L) \mapsto u_\eta \in U, \qquad \eta \in L_0^2(0,L) \mapsto \ell(\eta) \in \mathbb{R}$$

are linear and continuous.

Proof. i) *Existence.* From the Proposition 1, there exist $u_1, u_2, u_3 \in U$ solutions of

$$\begin{array}{rcll} a_S(u_1,\psi) & = & \langle f^S,\psi\rangle_{U',U} & \forall \psi \in U \\ a_S(u_2,\psi) & = & \int_0^L \eta(x_1)\,\psi(x_1)\,dx_1 & \forall \psi \in U \\ a_S(u_3,\psi) & = & \int_0^L \psi(x_1)\,dx_1 & \forall \psi \in U \end{array}$$

From the third equation and using that a_S is elliptic, we obtain

$$0 < a_S(u_3,u_3) = \int_0^L u_3(x_1)\,dx_1.$$

We search $c \in \mathbb{R}$ and $u = u_1 + u_2 + c \cdot u_3$ such that $\int_0^L u(x_1)\,dx_1 = 0$ or equivalently

$$c = -\frac{\int_0^L (u_1+u_2)dx_1}{\int_0^L u_3 dx_1}.$$

Uniqueness. Let $u_i, c_i, i = 1,2$ be two solutions of (12), such that $\int_0^L u_i\,dx_1 = 0$. By subtracting, we obtain

$$a_S(u_1-u_2,\psi) = (c_1-c_2)\int_0^L \psi(x_1)\,dx_1, \quad \forall \psi \in U$$

and after the substitution $\psi = u_1 - u_2$ it follows

$$a_S(u_1-u_2,u_1-u_2) = (c_1-c_2)\int_0^L (u_1-u_2)dx_1.$$

But $\int_0^L (u_1-u_2)dx_1 = 0$, then $a_S(u_1-u_2,u_1-u_2) = 0$ and consequently $u_1 = u_2$.

It follows that

$$0 = (c_1-c_2)\int_0^L \psi(x_1)\,dx_1, \quad \forall \psi \in U$$

then $c_1 = c_2$.

ii) From (13) and (14), we obtain that $u_0 + u_\eta \in U$ such that $\int_0^L u_0 + u_\eta\,dx_1 = 0$ and $c_0 + \ell(\eta) \in \mathbb{R}$ are solutions of

$$a_S(u_0+u_\eta,\psi) = \int_0^L (\eta(x_1) + c_0 + \ell(\eta))\,\psi(x_1)\,dx_1 + \langle f^S,\psi\rangle_{U',U} \quad \forall \psi \in U.$$

From the uniqueness proved at the point i), it follows that $u = u_0 + u_\eta$ and $c = c_0 + \ell(\eta)$.

It is easy to see that the applications $\eta \mapsto u_\eta$ and $\eta \mapsto \ell(\eta)$ are linear. It remains to prove the continuity.

We replace $\psi = u_\eta$ in (14) and using $\int_0^L u_\eta dx_1 = 0$, we obtain

$$a_S(u_\eta,u_\eta) = \int_0^L (\eta(x_1) + \ell(\eta))\,u_\eta(x_1)\,dx_1 = \int_0^L \eta(x_1)\,u_\eta(x_1)\,dx_1.$$

But a_S is elliptic and using the Cauchy-Schwartz inequality, we have

$$\|u_\eta\|_U^2 \leq C\|\eta\|_{L^2(0,L)}\|u_\eta\|_{L^2(0,L)} \leq C\|\eta\|_{L^2(0,L)}\|u_\eta\|_U$$

which proves the continuity of $\eta \mapsto u_\eta$.

From (14), we have

$$\ell(\eta)\int_0^L \psi\, dx_1 = a_S(u_\eta,\psi) - \int_0^L \eta\psi\, dx_1, \quad \forall\psi \in U.$$

We take $\psi_0 \in U$ such that $\int_0^L \psi_0 dx_1 > 0$ in the above equality. From the continuity of a_S, $\eta \mapsto u_\eta$ and using the Cauchy-Schwartz inequality, we obtain that $\eta \mapsto \ell(\eta)$ is continuous. □

Remark 2 *We obtain a displacement u such that $\int_0^L u\, dx_1 = 0$ if and only if the forces acting on the interface have the form $\eta + c_0 + \ell(\eta)$, where $\eta \in L_0^2(0,L)$.*

In order to obtain a connected fluid domain, we must impose some condition on f^S and η.

Let us denote by $\mathcal{S} : L^2(0,L) \to U$ the map

$$\mathcal{S}(\eta) = u, \tag{15}$$

where u is the unique solution of (11).

We define the admissible set for the forces induced by the fluid

$$\mathcal{F}_{ad} = \mathcal{S}^{-1}(\mathcal{U}_{ad}).$$

Let $u_0 \in U$, such that $\int_0^L u_0\, dx_1 = 0$ and $c_0 \in \mathbb{R}$ solutions of (13). We assume that

$$C_1\|c_0\|_{L^2(0,L)} + C_2\left\|f^S\right\|_{U'} < H - e$$

consequently $\|u_0\|_{L^\infty(0,L)} < H - e$.

Proposition 3 *i) The set $\mathcal{F}_{ad}$ is convex and closed in $L^2(0,L)$.*

ii) If $\|u_0\|_{L^\infty(0,L)} < H - e$, then $\mathcal{F}_{ad}$ is non empty.

Proof. i) The set $\mathcal{U}_{ad}$ is convex and closed in U. The application $\mathcal{S}$ is continuous and affine. Consequently, $\mathcal{F}_{ad}$ is convex and closed.

ii) We use the same notations as in the Proposition 2 part *ii*). From the continuity at $\eta = 0$ of the linear function $\eta \mapsto \ell(\eta)$, for small $\|\eta\|_{L^2(0,L)}$ we obtain $\|u_\eta\|_{L^\infty(0,L)} < H - e - \|u_0\|_{L^\infty(0,L)}$. So, if we set $u = u_0 + u_\eta$, we have

$$\begin{aligned}\|u\|_{L^\infty(0,L)} &\leq \|u_0\|_{L^\infty(0,L)} + \|u_\eta\|_{L^\infty(0,L)} \\ &< \|u_0\|_{L^\infty(0,L)} + H - e - \|u_0\|_{L^\infty(0,L)} = H - e,\end{aligned}$$

which implies that $H + u(x_1) \geq e$, $\forall x_1 \in [0,L]$. From the Proposition 2 we have that $u = \mathcal{S}(\eta + c_0 + \ell(\eta))$ verifies $\int_0^L u(x_1)\, dx_1 = 0$. Consequently, for small $\|\eta\|_{L^2(0,L)}$, we have $\eta + c_0 + \ell(\eta) \in \mathcal{F}_{ad}$. □

4 Mixed formulation in variable fluid domain

For each $\eta \in F_{ad}$, let u be the solution of the equation (11) and let Ω_u^F be the domain occupied by the fluid.

In view of the properties of the inclusion $H_0^2(0,L)$ in $\mathcal{C}^1(0,L)$ then the elastic boundary Γ_u is Lipschitz, so we can define the trace space $H^{1/2}(\Gamma_u)$. Moreover, from a classical result Theorem 2 in Vol. 6, p. 652 [10], the trace function mapping $H^1\left(\Omega_u^F\right)$ into $H^{1/2}(\Gamma_u)$ is continuous and onto.

In order to establish the variational formulation and the model for the u-dependent problem in the u-dependent fluid domain let us consider the following Hilbert spaces:

$$\begin{aligned} W_u &= \left\{w \in \left(H^1\left(\Omega_u^F\right)\right)^2;\ w_1 = 0 \text{ on } \partial\Omega_u^F, w_2 = 0 \text{ on } \overline{\Sigma}\right\}, \\ Q_u &= L^2\left(\Omega_u^F\right). \end{aligned}$$

We introduce in $\left(H^1\left(\Omega_u^F\right)\right)^2$ the divergence operator

$$\operatorname{div} w = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2}, \quad w = (w_1, w_2) \in \left(H^1\left(\Omega_u^F\right)\right)^2.$$

Next straightforward lemma states an important property of this operator.

Lemma 1 *For all u in $\mathcal{U}_{ad}$, the operator* div *mapping W_u into Q_u is onto.*

This result is standard for the homogenous Dirichlet boundary condition on the $\partial\Omega_u^F$.[19]

For the mixed boundary condition (Dirichlet on Σ and Neumann on Γ_u) and for an exterior domain (the complement of a compact set), the proof of this kind of result could be found in [32]. The proof remains valid in our case when the domain is bounded.

We denote by $\mu > 0$ the viscosity of the fluid and by $\epsilon(v) = (\epsilon_{ij}(v))_{1\le i,j\le 2}$ the symmetric part of the deformation rate tensor, where $\epsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$. Next, let us consider the maps

$$\left\{ \begin{array}{lll} a_F : U \times W_u \times W_u & \to & \mathbb{R} \\ \quad (u, v, w) & \mapsto & a_F(u, v, w) = 2\mu \displaystyle\sum_{i,j=1}^{2} \int_{\Omega_u^F} \epsilon_{ij}(v)\epsilon_{ij}(w)\, dx \end{array} \right. \tag{16}$$

and

$$\left\{ \begin{array}{lll} b_F : U \times W_u \times Q_u & \to & \mathbb{R} \\ \quad (u, w, q) & \mapsto & b_F(u, w, q) = -\displaystyle\int_{\Omega_u^F} (\operatorname{div} w)\, q\, dx. \end{array} \right. \tag{17}$$

The properties of the previous maps lead to the existence and uniqueness result [19]:

Proposition 4 *For all u in $\mathcal{U}_{ad}$ and λ in $L^2(\Gamma_u)$, the problem:*
Find $(v,\ p) \in W_u \times Q_u$ such that

$$\left\{ \begin{array}{lcl} a_F(u,v,w) + b_F(u,w,p) & = & \displaystyle\sum_{i=1}^{2} \int_{\Omega_u^F} f_i^F w_i \, dx + \int_{\Gamma_u} \lambda w_2 \, d\sigma, \ \forall w \in W_u \\ b_F(u,v,q) & = & 0, \ \forall q \in Q_u \end{array} \right. \tag{18}$$

has a unique solution.

Remark 3 *The system (18) represents a mixed formulation for the Stokes equations:*

$$\left\{ \begin{array}{rclcl} -\mu\Delta v + \nabla p & = & f^F & in & \Omega_u^F \\ \text{div } v & = & 0 & in & \Omega_u^F \\ v & = & 0 & on & \Sigma \\ \left(\sigma^F n\right) \cdot e_2 & = & \lambda & on & \Gamma_u \\ v_1 & = & 0 & on & \Gamma_u \end{array} \right.$$

where μ is the viscosity of the fluid, v and p represent the velocity and the pressure of the fluid, $f^F = \left(f_1^F, f_2^F\right)^T \in \mathbb{R}^2$ are the gravity forces, $\sigma^F = -pI + 2\mu\epsilon(v)$ is the stress tensor of the fluid, n is the unit outward normal vector to Γ_u, $e_2 = (0,1)^T$ is the unit vector in the x_2 direction, λ is the vertical component of the surface forces on the elastic boundary Γ_u. We have a Dirichlet homogeneous boundary condition on the rigid boundary Σ and on the elastic boundary Γ_u we have a Neumann and a Dirichlet boundary conditions.

The equilibrium of the physical situation, corresponding to a fluid which occupies a two-dimensional region whose boundary contains an elastic part, is based on the balance of velocity and normal forces in that boundary. In our approach to this particular fluid-structure model both balances are obtained in an optimal control problem setting. One of the first difficulties of this formulation is the u-dependence of the fluid domain. To overcome this problem in next section we propose an equivalent mixed formulation problem in a fixed domain but with u-dependent coefficients.

5 Mixed formulation for the fluid equations in a fixed domain

In order to obtain the mixed formulation for the fluid equations in a fixed domain, the arbitrary lagrangian eulerian coordinates have been used. For this formulation in a fixed domain we obtain the existence of the solution.

For each $u \in U$ be given, let us consider the following one-to-one continuous differentiable transformation:

$$T_u : \overline{\Omega_0^F} \to \overline{\Omega_u^F}, \quad (\widehat{x}_1, \widehat{x}_2) \mapsto T_u(\widehat{x}_1, \widehat{x}_2) = \left(\widehat{x}_1, \frac{H + u(\widehat{x}_1)}{H} \widehat{x}_2\right) \tag{19}$$

which admits the continuous differentiable inverse mapping

$$T_u^{-1} : \overline{\Omega_u^F} \to \overline{\Omega_0^F}, \quad (x_1, x_2) \mapsto T_u^{-1}(x_1, x_2) = \left(x_1, \frac{Hx_2}{H + u(x_1)} \right) \tag{20}$$

and verifies that $T_u\left(\Omega_0^F\right) = \Omega_u^F$, $T_u(\Gamma_0) = \Gamma_u$ and $T_u(\widehat{x}) = \widehat{x}$, $\forall \widehat{x} \in \Sigma$. We set $x = T_u(\widehat{x})$ for each $x = (x_1, x_2) \in \Omega_u^F$ and $\widehat{x} = (\widehat{x}_1, \widehat{x}_2) \in \Omega_0^F$. We note $\sigma = T_u(\widehat{\sigma})$ for each $\sigma \in \Gamma_u$ and $\widehat{\sigma} \in \Gamma_0$.

Moreover, we denote by

$$\begin{aligned} \nabla T_u(\widehat{x}) &= \begin{pmatrix} 1 & 0 \\ \frac{u'(\widehat{x}_1)}{H}\widehat{x}_2 & \frac{H+u(\widehat{x}_1)}{H} \end{pmatrix} \\ \nabla\left(T_u^{-1}\right)(x) &= \begin{pmatrix} 1 & 0 \\ \frac{-u'(x_1)Hx_2}{(H+u(x_1))^2} & \frac{H}{H+u(x_1)} \end{pmatrix} \end{aligned}$$

the jacobian matrices of the transformations T_u and T_u^{-1} respectively. As usual for a given square matrix A, we denote by $\det(A)$, A^{-1}, A^T, $\operatorname{cof}(A)$ the determinant, the inverse, the transpose and the cofactor matrix, respectively. We have

$$(\nabla T_u)^{-1}(\widehat{x}) = \nabla\left(T_u^{-1}\right)(x) = \nabla\left(T_u^{-1}\right)(T_u(\widehat{x}))$$

and

$$\operatorname{cof}(\nabla T_u(\widehat{x})) = \det(\nabla T_u(\widehat{x}))\left((\nabla T_u(\widehat{x}))^{-1}\right)^T.$$

Associated with the transformation T_u we state the following useful lemma.

Lemma 2 *We have:*

1. *A function ϕ belongs to $L^1\left(\Omega_u^F\right)$ if and only if the function $\widehat{\phi} = \phi \circ T_u$ belongs to $L^1\left(\Omega_0^F\right)$. Moreover, in this case we have*

$$\int_{\Omega_u^F} \phi(x)\, dx = \int_{\Omega_0^F} \widehat{\phi}(\widehat{x}) \det(\nabla T_u(\widehat{x}))\, d\widehat{x}. \tag{21}$$

2. *A function ϕ belongs to $L^1(\Gamma_u)$ if and only if the function $\widehat{\phi} = \phi \circ T_u$ belongs to $L^1(\Gamma_0)$. Moreover, in this case we have*

$$\int_{\Gamma_u} \phi(\sigma)\, d\sigma = \int_{\Gamma_0} \widehat{\phi}(\widehat{\sigma})\, \widehat{\omega}_u(\widehat{\sigma})\, d\widehat{\sigma} \tag{22}$$

where $\widehat{\omega}_u(\widehat{\sigma})$ is given by

$$\widehat{\omega}_u(\widehat{\sigma}) = \|\operatorname{cof}(\nabla T_u(\widehat{\sigma}))\, \widehat{n}(\widehat{\sigma})\|_{\mathbb{R}^2} \tag{23}$$

with $\widehat{n}(\widehat{\sigma})$ being the unit outward normal vector to Γ_0 in $\widehat{\sigma}$.

3. *A function ϕ belongs to $H^1\left(\Omega_u^F\right)$ if and only if the function $\widehat{\phi} = \phi \circ T_u$ belongs to $H^1\left(\Omega_0^F\right)$. Moreover, we have*

$$\left(\begin{array}{c} \frac{\partial \phi}{\partial x_1}(x) \\ \frac{\partial \phi}{\partial x_2}(x) \end{array} \right) = \left((\nabla T_u)^{-1}(\widehat{x}) \right)^T \left(\begin{array}{c} \frac{\partial \widehat{\phi}}{\partial \widehat{x}_1}(\widehat{x}) \\ \frac{\partial \widehat{\phi}}{\partial \widehat{x}_2}(\widehat{x}) \end{array} \right). \tag{24}$$

The first and second assertions of the above lemma follow from the well-known transport theorems in continuum mechanics.[20] The third part of the lemma is a consequence of basic results for Sobolev spaces [1] and the chain rule.

In our case, we have

$$\det\left(\nabla T_u(\widehat{x})\right) = \frac{H + u(\widehat{x}_1)}{H}, \quad \widehat{\omega}_u(\widehat{x}_1, H) = \sqrt{1 + \left(u'(\widehat{x}_1)\right)^2}.$$

We denote by

$$(\nabla T_u)^{-1}(\widehat{x}) = \left(\begin{array}{cc} 1 & 0 \\ \frac{-u'(\widehat{x}_1)\widehat{x}_2}{H+u(\widehat{x}_1)} & \frac{H}{H+u(\widehat{x}_1)} \end{array} \right) = \left(\begin{array}{cc} s_{11}(\widehat{x}) & s_{12}(\widehat{x}) \\ s_{21}(\widehat{x}) & s_{22}(\widehat{x}) \end{array} \right)$$

and as a consequence of the above Lemma, we have

$$\left(\begin{array}{cc} \frac{\partial v_1}{\partial x_1}(x) & \frac{\partial v_1}{\partial x_2}(x) \\ \frac{\partial v_2}{\partial x_1}(x) & \frac{\partial v_2}{\partial x_2}(x) \end{array} \right) = \left(\begin{array}{cc} \frac{\partial \widehat{v}_1}{\partial \widehat{x}_1}(\widehat{x}) & \frac{\partial \widehat{v}_1}{\partial \widehat{x}_2}(\widehat{x}) \\ \frac{\partial \widehat{v}_2}{\partial \widehat{x}_1}(\widehat{x}) & \frac{\partial \widehat{v}_2}{\partial \widehat{x}_2}(\widehat{x}) \end{array} \right) \left(\begin{array}{cc} s_{11}(\widehat{x}) & s_{12}(\widehat{x}) \\ s_{21}(\widehat{x}) & s_{22}(\widehat{x}) \end{array} \right).$$

In order to pose the variational formulation in the reference configuration let us consider the following Hilbert spaces:

$$\begin{aligned} \widehat{W} &= \left\{ \widehat{w} \in \left(H^1\left(\Omega_0^F\right)\right)^2;\ \ \widehat{w}_1 = 0 \text{ on } \partial\Omega_0^F, \widehat{w}_2 = 0 \text{ on } \overline{\Sigma} \right\} \\ \widehat{Q} &= L^2\left(\Omega_0^F\right) \end{aligned}$$

equipped with their usual inner products.

We introduce the forms

$$\widehat{a}_F : \mathcal{U}_{ad} \times \widehat{W} \times \widehat{W} \to \mathbb{R} \qquad \widehat{b}_F : \mathcal{U}_{ad} \times \widehat{W} \times \widehat{Q} \to \mathbb{R}$$

defined by

$$\begin{aligned} &\widehat{a}_F(u, \widehat{v}, \widehat{w}) = 2\mu \int_{\Omega_0^F} \left[\left(\tfrac{\partial \widehat{v}_1}{\partial \widehat{x}_1} s_{11} + \tfrac{\partial \widehat{v}_1}{\partial \widehat{x}_2} s_{21} \right) \left(\tfrac{\partial \widehat{w}_1}{\partial \widehat{x}_1} s_{11} + \tfrac{\partial \widehat{w}_1}{\partial \widehat{x}_2} s_{21} \right) \right. \\ &+ \tfrac{1}{2} \left(\tfrac{\partial \widehat{v}_1}{\partial \widehat{x}_1} s_{12} + \tfrac{\partial \widehat{v}_1}{\partial \widehat{x}_2} s_{22} + \tfrac{\partial \widehat{v}_2}{\partial \widehat{x}_1} s_{11} + \tfrac{\partial \widehat{v}_2}{\partial \widehat{x}_2} s_{21} \right) \left(\tfrac{\partial \widehat{w}_1}{\partial \widehat{x}_1} s_{12} + \tfrac{\partial \widehat{w}_1}{\partial \widehat{x}_2} s_{22} + \tfrac{\partial \widehat{w}_2}{\partial \widehat{x}_1} s_{11} + \tfrac{\partial \widehat{w}_2}{\partial \widehat{x}_2} s_{21} \right) \\ &+ \left. \left(\tfrac{\partial \widehat{v}_2}{\partial \widehat{x}_1} s_{12} + \tfrac{\partial \widehat{v}_2}{\partial \widehat{x}_2} s_{22} \right) \left(\tfrac{\partial \widehat{w}_2}{\partial \widehat{x}_1} s_{12} + \tfrac{\partial \widehat{w}_2}{\partial \widehat{x}_2} s_{22} \right) \right] \det\left(\nabla T_u(\widehat{x})\right) d\widehat{x} \\ &= 2\mu \sum_{i,j,k,\ell=1}^{2} \int_{\Omega_0^F} a_{k,\ell}^{i,j}(u, \widehat{x}) \frac{\partial \widehat{v}_i}{\partial \widehat{x}_k} \frac{\partial \widehat{w}_j}{\partial \widehat{x}_\ell} d\widehat{x}. \end{aligned} \tag{25}$$

$$\begin{aligned}\widehat{b}_F\left(u,\widehat{w},\widehat{q}\right) &= -\int_{\Omega_0^F}\left(\frac{\partial\widehat{v}_1}{\partial\widehat{x}_1}s_{11}+\frac{\partial\widehat{v}_1}{\partial\widehat{x}_2}s_{21}+\frac{\partial\widehat{v}_2}{\partial\widehat{x}_1}s_{12}+\frac{\partial\widehat{v}_2}{\partial\widehat{x}_2}s_{22}\right)\widehat{q}\,\det\left(\nabla T_u\left(\widehat{x}\right)\right)d\widehat{x}\\ &= -\int_{\Omega_0^F}\left(\frac{\partial\widehat{v}_1}{\partial\widehat{x}_1}\frac{H+u(\widehat{x}_1)}{H}-\frac{\partial\widehat{v}_1}{\partial\widehat{x}_2}\frac{u'(\widehat{x}_1)\widehat{x}_2}{H}+\frac{\partial\widehat{v}_2}{\partial\widehat{x}_2}\right)\widehat{q}\,d\widehat{x}. \qquad (26)\end{aligned}$$

Let us consider $\widehat{f}^F(u)\in\widehat{W}'$ defined for all $\widehat{w}$ in $\widehat{W}$ by

$$\begin{aligned}\left\langle\widehat{f}^F(u),\widehat{w}\right\rangle_{\widehat{W}',\widehat{W}} &= \sum_{i=1}^{2}\int_{\Omega_0^F}f_i^F\,\widehat{w}_i\det\left(\nabla T_u\left(\widehat{x}\right)\right)\,d\widehat{x}\\ &= \sum_{i=1}^{2}\int_{\Omega_0^F}f_i^F\,\widehat{w}_i\frac{H+u(\widehat{x}_1)}{H}\,d\widehat{x}.\end{aligned}$$

Proposition 5 *For all u in $\mathcal{U}_{ad}$ and $\widehat{\lambda}$ in $L^2(\Gamma_0)$, the problem:*
Find $(\widehat{v},\ \widehat{p})\in\widehat{W}\times\widehat{Q}$ such that

$$\left\{\begin{array}{lcl}\widehat{a}_F\left(u,\widehat{v},\widehat{w}\right)+\widehat{b}_F\left(u,\widehat{w},\widehat{p}\right) &=& \left\langle\widehat{f}^F(u),\widehat{w}\right\rangle+\int_{\Gamma_0}\widehat{\lambda}\widehat{w}_2\,d\widehat{\sigma},\forall\widehat{w}\in\widehat{W}\\ \widehat{b}_F\left(u,\widehat{v},\widehat{q}\right) &=& 0,\forall\widehat{q}\in\widehat{Q}\end{array}\right. \qquad (27)$$

has a unique solution.

The problem (27) is obtained from (18) and reversely by using the one-to-one transformations T_u and T_u^{-1}. We have $\widehat{v}=v\circ T_u$, $\widehat{p}=p\circ T_u$ and $\widehat{\lambda}=\widehat{\omega}_u(\lambda\circ T_u)$, where $\widehat{\omega}_u$ is given by the formula (23). Therefore, the Proposition 5 is a consequence of the Proposition 4.

6 Optimal control setting

Let us consider the space $\widehat{M}=L^2(\Gamma_0)$. Next, we introduce the two linear and bounded operators

$$A_S:U\to U' \qquad C_S:\widehat{M}\to U'$$

defined by

$$\begin{aligned}\left\langle A_S\phi,\psi\right\rangle_{U',U} &= a_S(\phi,\psi) \qquad \forall\phi,\psi\in U \qquad (28)\\ \left\langle C_S\widehat{\lambda},\psi\right\rangle_{U',U} &= \int_0^L\widehat{\lambda}\ (x_1,H)\,\psi\,dx_1 \qquad \forall\widehat{\lambda}\in\widehat{M},\forall\psi\in U\end{aligned}$$

where a_S is defined by (10).

In terms of the operators defined by (28), equation (11) is posed in the form

$$A_S\,u\left(\widehat{\lambda}\right)=-C_S\,\widehat{\lambda}+f^S$$

which points out that the displacement of the structure $u\left(\widehat{\lambda}\right)$ depends on the forces $\widehat{\lambda}$.

For each u in $\mathcal{U}_{ad}$, there exist three linear bounded operators

$$A_F(u) : \widehat{W} \to \widehat{W}', \qquad B_F(u) : \widehat{W} \to \widehat{Q}', \qquad C_F : \widehat{M} \to \widehat{W}'$$

given by

$$\begin{aligned}
\langle A_F(u)\widehat{v}, \widehat{w}\rangle_{\widehat{W}',\widehat{W}} &= \widehat{a}_F(u, \widehat{v}, \widehat{w}), \quad \forall \widehat{v}, \widehat{w} \in \widehat{W} \\
\langle B_F(u)\widehat{w}, \widehat{q}\rangle_{\widehat{Q}',\widehat{Q}} &= \widehat{b}_F(u, \widehat{w}, \widehat{q}), \quad \forall \widehat{w} \in \widehat{W}, \forall \widehat{q} \in \widehat{Q} \\
\left\langle C_F\widehat{\lambda}, w\right\rangle_{\widehat{W}',\widehat{W}} &= \int_{\Gamma_0} \widehat{\lambda}\widehat{w}_2 \, d\widehat{\sigma}, \quad \forall \widehat{\lambda} \in \widehat{M}, \forall \widehat{w} \in \widehat{W}.
\end{aligned} \tag{29}$$

So, the system (27) can be rewritten with operator notation in the form:

For $u \in \mathcal{U}_{ad}$ and $\widehat{\lambda} \in \widehat{M}$ given, find $\widehat{v}\left(u, \widehat{\lambda}\right) \in \widehat{W}$ and $\widehat{p}\left(u, \widehat{\lambda}\right) \in \widehat{Q}$ such that

$$\left\{ \begin{array}{lcll}
A_F(u)\widehat{v}\left(u, \widehat{\lambda}\right) + B_F^*(u)\widehat{p}\left(u, \widehat{\lambda}\right) &=& \widehat{f}^F(u) + C_F\widehat{\lambda} & \text{in } \widehat{W}' \\
B_F(u)\widehat{v}\left(u, \widehat{\lambda}\right) &=& 0 & \text{in } \widehat{Q}'
\end{array} \right. \tag{30}$$

or in an equivalent matrix notation as

$$\begin{pmatrix} A_F(u) & B_F^*(u) \\ B_F(u) & 0 \end{pmatrix} \begin{pmatrix} \widehat{v}\left(u, \widehat{\lambda}\right) \\ \widehat{p}\left(u, \widehat{\lambda}\right) \end{pmatrix} = \begin{pmatrix} \widehat{f}^F(u) + C_F\widehat{\lambda} \\ 0 \end{pmatrix} \tag{31}$$

where $B_F^*(u)$ is the adjoint operator of $B_F(u)$.

In the next paragraph the fluid-structure coupled problem will be modeled by an optimal control system.

For each $\widehat{v} \in \widehat{W}$ we denote by $\widehat{v}_{|\Gamma_0}$ the trace on Γ_0 of $\widehat{v}$ and we denote by $\|\cdot\|_{0,\Gamma_0}$ the usual norm in $L^2(\Gamma_0)$. We denote by $J : \widehat{W} \to \mathbb{R}$, the function defined by

$$J(\widehat{w}) = \frac{1}{2} \left\| \widehat{w}_{2|\Gamma_0} \right\|_{0,\Gamma_0}^2 .$$

Moreover, let $j : \mathcal{F}_{ad} \to \mathbb{R}$ be the function defined by

$$j\left(\widehat{\lambda}\right) = J\left(\widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)\right). \tag{32}$$

We pose the following optimal control problem $(\mathcal{P})$:

$$\inf j\left(\widehat{\lambda}\right)$$

subject to the conditions:

1. $\widehat{\lambda} \in \mathcal{F}_{ad}$
2. $u\left(\widehat{\lambda}\right) \in \mathcal{U}_{ad}$ such that

$$A_S\, u\left(\widehat{\lambda}\right) = -C_S\, \widehat{\lambda} + f^S \tag{33}$$

3. $\widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \in \widehat{W}, \quad \widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \in \widehat{Q}$ such that

$$\begin{pmatrix} A_F\left(u\left(\widehat{\lambda}\right)\right) & B_F^*\left(u\left(\widehat{\lambda}\right)\right) \\ B_F\left(u\left(\widehat{\lambda}\right)\right) & 0 \end{pmatrix} \begin{pmatrix} \widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \\ \widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \end{pmatrix} = \begin{pmatrix} \widehat{f}^F(u) + C_F \widehat{\lambda} \\ 0 \end{pmatrix}. \tag{34}$$

Therefore, the previous formulation corresponds to an optimal control problem with Neumann like boundary control $(\widehat{\lambda})$ and Dirichlet like boundary observation $(\widehat{v}_{2|\Gamma_0})$. Moreover, the control appears also in the coefficients of the fluid equations (34) as it happens in some optimal design problems.[37], [41] The condition $\widehat{\lambda} \in \mathcal{F}_{ad}$ represents the control constraint, while the state constraint is given by the fact that $u\left(\widehat{\lambda}\right) \in \mathcal{U}_{ad}$.

This mathematical formulation provides an interesting tool for the numerical approximation of the *a priori* fluid-structure coupled problem in an uncoupled way. That is, the structure equations represented by the first two conditions and the fluid equations (34) can be solved separately in an iterative process.

As we mentioned in the second section, on the interface we have two boundary conditions: equality of the fluid and structure velocities (which is a Dirichlet like boundary condition) and equality of the stresses (which is a Neumann like boundary condition). In our approach we pursue both coupling conditions in the iterative algorithm:

- Step 1: We start with a guess for the forces $\widehat{\lambda}$ on the interface.
- Step 2: The displacement $u\left(\widehat{\lambda}\right)$ of the structure can be computed by (33).
- Step 3: Once the coefficients of the equations (34) have been obtained, we can compute the velocity and the pressure of the fluid as the solution of the weak mixed formulation on the fixed domain (34).
- Step 4: Update $\widehat{\lambda}$ in order to minimize the cost function j.

Remark 4 *As we use the value $-\widehat{\lambda}$ for the forces on Γ_0 in the equations of the structure and we take the value $\widehat{\lambda}$ in the equations of the fluid, the Neumann like boundary condition is strongly accomplished. The Dirichlet like boundary condition $\widehat{v}_{2|\Gamma_0} = 0$ is approached by a Least Square formulation posed in terms of the minimization problem*

$$\inf \frac{1}{2} \left\| \widehat{v}_{2|\Gamma_0} \right\|_{0,\Gamma_0}^2 .$$

7 Continuity of the cost function

In this subsection, we shall prove that the cost function j is continuous.

The cost function is the composition of the following functions:

$$\begin{aligned} \widehat{\lambda} \in \mathcal{F}_{ad} &\longmapsto u\left(\widehat{\lambda}\right) \in \mathcal{U}_{ad}, \\ \left(u, \widehat{\lambda}\right) \in \mathcal{U}_{ad} \times \widehat{M} &\longmapsto \widehat{v}\left(u, \widehat{\lambda}\right) \in \widehat{W}, \\ \widehat{w} \in \widehat{W} &\longmapsto J\left(\widehat{w}\right) \in \mathbb{R}. \end{aligned}$$

The first and the third are continuous, evidently. Next, by using the Implicit Function Theorem (see the Appendix), we shall prove that the second one is continuous too.

We define

$$\begin{aligned} \mathcal{U} = & \{u \in U; u(0) = u(L) = u'(0) = u'(L) = 0, \\ & H + u(x_1) > 0, \forall x_1 \in [0, L]\}, \end{aligned} \tag{35}$$

so that $\mathcal{U}_{ad} \subset \mathcal{U} \subset U$ and $\mathcal{U}$ is an open set of U.

Let us consider the function $h : \left(\widehat{M} \times \mathcal{U}\right) \times \left(\widehat{W} \times \widehat{Q}\right) \to \widehat{W}' \times \widehat{Q}'$ defined by

$$h\left(\left(\widehat{\mu}, u\right), \left(\widehat{w}, \widehat{q}\right)\right) = \left(A_F(u)\,\widehat{w} + B_F^*(u)\,\widehat{q} - \widehat{f}^F(u) - C_F\widehat{\mu}, B_F(u)\,\widehat{w}\right).$$

Next we apply Theorem 4 (see the Appendix) for the case

$$\begin{gathered} X = \widehat{M} \times U,\ Y = Z = \widehat{W} \times \widehat{Q},\ G = \widehat{M} \times \mathcal{U} \times \widehat{W} \times \widehat{Q}, \\ x_0 = \left(\widehat{\lambda}, u\left(\widehat{\lambda}\right)\right),\ y_0 = \left(\widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right), \widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)\right), \\ x = \left(\widehat{\mu}, u\right),\ y = \left(\widehat{w}, \widehat{q}\right). \end{gathered}$$

We have that $h(x_0, y_0) = 0$. According to the Proposition 5 and in view of the identities (30) and (31), we have that

$$\frac{\partial h}{\partial y}\left(\left(\widehat{\mu}, u\right), \left(\widehat{w}, \widehat{q}\right)\right) = \begin{pmatrix} A_F(u) & B_F^*(u) \\ B_F(u) & 0 \end{pmatrix} \in \mathcal{L}\left(\widehat{W} \times \widehat{Q}, \widehat{W}' \times \widehat{Q}'\right)$$

is invertible.

In view of the Remark 6 (see the Appendix), it remains to verify that h and $\frac{\partial h}{\partial y}$ are continuous in (x_0, y_0).

Proposition 6 *Let $\overline{u}$ be in $\mathcal{U}_{ad}$. We have*

$$\lim_{\|u - \overline{u}\|_U \to 0} \|A_F(u) - A_F(\overline{u})\|_{L\left(\widehat{W}, \widehat{W}'\right)} = 0, \tag{36}$$

$$\lim_{\|u - \overline{u}\|_U \to 0} \|B_F(u) - B_F(\overline{u})\|_{L\left(\widehat{W}, \widehat{Q}'\right)} = 0, \tag{37}$$

$$\lim_{\|u - \overline{u}\|_U \to 0} \|B_F^*(u) - B_F^*(\overline{u})\|_{L\left(\widehat{Q}, \widehat{W}'\right)} = 0 \tag{38}$$

where $\|\cdot\|_U$ is the norm of the Sobolev space $U = H_0^2(0, L)$.

Proof. We have that

$$\langle A_F(u)\,\widehat{v},\widehat{w}\rangle = \widehat{a}_F(u,\widehat{v},\widehat{w})$$

where $\widehat{a}_F$ is defined by (25).

Next, by using the elementary integral calculus results, we obtain

$$\begin{aligned} |u(\widehat{x}_1)| &= \left|\int_0^{\widehat{x}_1} u'(s)\,ds\right| \le \int_0^{\widehat{x}_1} |u'(s)|\,ds \\ &\le \int_0^L |u'(s)|\,ds \le \left(\int_0^L |u'(s)|^2\,ds\right)^{1/2} \le \|u\|_U \end{aligned} \tag{39}$$

and analogously

$$\begin{aligned} |u'(\widehat{x}_1)| &= \left|\int_0^{\widehat{x}_1} u''(s)\,ds\right| \le \int_0^{\widehat{x}_1} |u''(s)|\,ds \\ &\le \int_0^L |u''(s)|\,ds \le \left(\int_0^L |u''(s)|^2\,ds\right)^{1/2} \le \|u\|_U\,. \end{aligned} \tag{40}$$

Since the coefficients of the bilinear form $\widehat{a}_F(u,\cdot,\cdot)$ are continuous with respect to u, u' and thanks to above inequalities, we obtain that there exists a constant $C_1\left(\Omega_0^F\right)$ depending only upon the shape of the domain Ω_0^F, such that for all $\widehat{v}$ and $\widehat{w}$ in $\widehat{W}$, we have

$$\widehat{a}_F(u-\overline{u},\widehat{v},\widehat{w}) \le C_1\left(\Omega_0^F\right)\|u-\overline{u}\|_U\,\|\widehat{v}\|_{\widehat{W}}\,\|\widehat{w}\|_{\widehat{W}}\,.$$

It was essential for obtaining the above estimation the fact that the domain Ω_0^F is bounded!

It follows that

$$\begin{aligned} &\|A_F(u)-A_F(\overline{u})\|_{L\left(\widehat{W},\widehat{W}'\right)} \overset{def}{=} \sup_{\|\widehat{v}\|_{\widehat{W}}\le 1,\|\widehat{w}\|_{\widehat{W}}\le 1} \langle (A_F(u)-A_F(\overline{u}))\,\widehat{v},\widehat{w}\rangle \\ &= \sup_{\|\widehat{v}\|_{\widehat{W}}\le 1,\|\widehat{w}\|_{\widehat{W}}\le 1} \widehat{a}_F(u-\overline{u},\widehat{v},\widehat{w}) \le C_1\left(\Omega_0^F\right)\|u-\overline{u}\|_U\,. \end{aligned}$$

which proves the relation (36).

Analogously, we obtain the two other relations which complete the proof. □

Proposition 7 *The function*

$$u \in \mathcal{U}_{ad} \mapsto \widehat{f}^F(u) \in \widehat{W}'$$

is continuous.

Proof. We have

$$\begin{aligned}\left\|\widehat{f}^F(u)-\widehat{f}^F(\overline{u})\right\| &= \sup_{\|\widehat{w}\|\leq 1}\left|\sum_{i=1}^{2}\int_{\Omega_0^F}\widehat{f}_i^F(u-\overline{u})(\widehat{x}_1)\,\widehat{w}_i\,d\widehat{x}\right| \\ &\leq \|u-\overline{u}\|_U \sup_{\|\widehat{w}\|\leq 1}\left(\sum_{i=1}^{2}\int_{\Omega_0^F}\left|\widehat{f}_i^F\right|\left|\widehat{w}_i\right|d\widehat{x}\right) \\ &\leq \|u-\overline{u}\|_U\sqrt{\sum_{i=1}^{2}\int_{\Omega_0^F}\left|\widehat{f}_i^F\right|^2 d\widehat{x}}\end{aligned}$$

and the conclusion holds. □

Corollary 1 *The function* $\frac{\partial h}{\partial y}$ *from* G *to* $\mathcal{L}\left(\widehat{W}\times\widehat{Q},\widehat{W}'\times\widehat{Q}'\right)$ *is continuous on* G.

Corollary 2 *The functions*

$$\begin{aligned}(u,\widehat{w})\in\ \mathcal{U}_{ad}\times\widehat{W}&\longmapsto A_F(u)\,\widehat{w}\in\widehat{W}'\\(u,\widehat{w})\in\ \mathcal{U}_{ad}\times\widehat{W}&\longmapsto B_F(u)\,\widehat{w}\in\widehat{Q}'\\(u,\widehat{q})\in\ \mathcal{U}_{ad}\times\widehat{Q}&\longmapsto B_F^*(u)\,\widehat{q}\in\widehat{W}'\end{aligned}$$

are continuous.

Proof. Let $\overline{u}$ and $\widehat{v}$ be given in $\mathcal{U}_{ad}$ and $\widehat{W}$ respectively. We have

$$\begin{aligned}\|A_F(u)\,\widehat{w}-A_F(\overline{u})\,\widehat{v}\|_{\widehat{W}'} &\leq\\ \|A_F(u)\,\widehat{w}-A_F(\overline{u})\,\widehat{w}+A_F(\overline{u})\,\widehat{w}-A_F(\overline{u})\,\widehat{v}\|_{\widehat{W}'} &\leq\\ \|A_F(u)-A_F(\overline{u})\|_{L\left(\widehat{W},\widehat{W}'\right)}\|\widehat{w}\|_{\widehat{W}}+\|A_F(\overline{u})\|_{L\left(\widehat{W},\widehat{W}'\right)}\|\widehat{w}-\widehat{v}\|_{\widehat{W}}&.\end{aligned}$$

From Proposition 6, we have

$$\lim_{\|u-\overline{u}\|_U\to 0}\|A_F(u)-A_F(\overline{u})\|_{L\left(\widehat{W},\widehat{W}'\right)}=0.$$

Next, since $\|\widehat{w}-\widehat{v}\|_{\widehat{W}}\to 0$, we get that $\|\widehat{w}\|_{\widehat{W}}$ is bounded and the proof is complete. □

Corollary 3 *The function* h *from* G *to* $\widehat{W}'\times\widehat{Q}'$ *is continuous on* G.

All the hypotheses of the Theorem 4 (see the Appendix) hold, so the implicit function $\theta:\widehat{M}\times\mathcal{U}\to\widehat{W}\times\widehat{Q}$ given by

$$\theta(\widehat{\mu},u)=(\widehat{v}(u,\widehat{\mu}),\widehat{p}(u,\widehat{\mu}))$$

is continuous in $\left(\widehat{\lambda},u\left(\widehat{\lambda}\right)\right)$. Moreover, this result holds for each $\widehat{\lambda}\in\widehat{M}$, such that $u\left(\widehat{\lambda}\right)\in\mathcal{U}$.

Therefore, we obtain that the cost function j defined in (32) is continuous on $\mathcal{F}_{ad}$, since it is the composition of three continuous functions.

8 Differentiability of the cost function

In this section we analyze the differentiability of the cost function as well as the expression of its gradient. We use the method of deformation of domains.[31], [7], [9]

In the four following lemmas, the differentiability of intermediate functions is established. Moreover, the analytic formula for their derivatives is obtained.

We follow the notations introduced in the previous sections.

Lemma 3 *The function* $J : \widehat{W} \to \mathbb{R}$ *defined by*

$$J(\widehat{w}) = \frac{1}{2}\|\widehat{w}_2\|^2_{0,\Gamma_0}$$

is Fréchet differentiable and

$$J'(\widehat{v})\,\widehat{w} = \int_{\Gamma_0} \widehat{v}_2 \widehat{w}_2 \, d\widehat{\sigma}.$$

Proof. The function $\widehat{w} \longmapsto \int_{\Gamma_0} \widehat{v}_2\widehat{w}_2\, d\widehat{\sigma}$ is linear and continuous, evidently. We shall use the definition of the Fréchet differentiability detailed in the Appendix.

$$\begin{aligned} &\lim_{\widehat{w}\to 0} \frac{\left|\frac{1}{2}\|\widehat{v}_2+\widehat{w}_2\|^2_{0,\Gamma_0} - \frac{1}{2}\|\widehat{v}_2\|^2_{0,\Gamma_0} - \int_{\Gamma_0}\widehat{v}_2\widehat{w}_2\, d\widehat{\sigma}\right|}{\|\widehat{w}\|_{1,\Omega_0^F}} \\ &= \lim_{\widehat{w}\to 0} \frac{\|\widehat{w}_2\|^2_{0,\Gamma_0}}{2\|\widehat{w}\|_{1,\Omega_0^F}} = \lim_{\widehat{w}\to 0} \frac{\|\widehat{w}_2\|_{0,\Gamma_0}}{\|\widehat{w}\|_{1,\Omega_0^F}} \frac{\|\widehat{w}_2\|_{0,\Gamma_0}}{2}. \end{aligned}$$

Since $\|\widehat{w}_2\|_{0,\Gamma_0} \le \|\widehat{w}\|_{0,\Gamma_0}$ and from the continuity of the trace operator defined on $H^1\left(\Omega_0^F\right)$, we have

$$\frac{\|\widehat{w}_2\|_{0,\Gamma_0}}{\|\widehat{w}\|_{1,\Omega_0^F}} \le \frac{\|\widehat{w}\|_{0,\Gamma_0}}{\|\widehat{w}\|_{1,\Omega_0^F}} \le const.$$

so the above limit is 0. □

Lemma 4 *Let* $\widehat{v}$, $\widehat{w}$ *be given in* $\widehat{W}$ *and* $\widehat{q}$ *in* $\widehat{Q}$. *Then the functions from* $\mathcal{U}_{ad}$ *to* $\mathbb{R}$ *defined by*

$$u \longmapsto \widehat{a}_F(u, \widehat{v}, \widehat{w})$$
$$u \longmapsto \widehat{b}_F(u, \widehat{w}, \widehat{q})$$

are Fréchet differentiable on $\mathcal{U}_{ad}$ *and the derivatives have the forms:*

$$\frac{\partial \widehat{a}_F}{\partial u}(u, \widehat{v}, \widehat{w})\,\psi = 2\mu \sum_{i,j,k,\ell=1}^{2} \int_{\Omega_0^F} \frac{\partial a^{i,j}_{k,\ell}}{\partial u}(u, \widehat{x})\,\psi \frac{\partial \widehat{v}_i}{\partial \widehat{x}_k}\frac{\partial \widehat{w}_j}{\partial \widehat{x}_\ell}\, d\widehat{x} \tag{41}$$

$$\frac{\partial \widehat{b}_F}{\partial u}(u, \widehat{w}, \widehat{q})\,\psi = -\int_{\Omega_0^F} \frac{\psi(\widehat{x}_1)}{H}\frac{\partial \widehat{w}_1}{\partial \widehat{x}_1}\widehat{q}\, d\widehat{x} + \int_{\Omega_0^F} \frac{\psi'(\widehat{x}_1)\,\widehat{x}_2}{H}\frac{\partial \widehat{w}_1}{\partial \widehat{x}_2}\widehat{q}\, d\widehat{x}. \tag{42}$$

Proof. In view of the identity (26), the function

$$u \mapsto \widehat{b}_F\left(u, \widehat{w}, \widehat{q}\right)$$

is affine. Using the inequalities (39) and (40), we get the continuity of this function. Consequently, it is Fréchet differentiable.

But, for a linear and continuous function

$$u \in U \mapsto f\left(u\right) \in \mathbb{R}$$

the Fréchet derivative has the form

$$f'\left(u\right)\psi = f\left(\psi\right), \qquad \forall \psi \in U.$$

The above identity gives (42).

Using the same method, we can get the Fréchet differentiability and derivatives for all the terms of $\widehat{a}_F\left(u, \widehat{v}, \widehat{w}\right)$ which are affine with respect to u.

The only remaining point concerns the differentiability of the function

$$u \longmapsto \int_{\Omega_0^F} a_{k,\ell}^{i,j}\left(u, \widehat{x}\right) \frac{\partial \widehat{v}_i}{\partial \widehat{x}_k} \frac{\partial \widehat{w}_j}{\partial \widehat{x}_\ell}\, d\widehat{x}.$$

where $u \in U \mapsto a_{k,\ell}^{i,j}\left(u, \cdot\right) \in L^\infty\left(\Omega_0^F\right)$ is nonlinear.

We apply the Theorem 7 concerning the differentiability of integrals with parameter (see the Appendix) in the case $\widehat{\Omega} \equiv \overline{\Omega}_0^F$ and

$$f\left(u, \widehat{x}\right) = a_{k,\ell}^{i,j}\left(u, \widehat{x}\right) \frac{\partial \widehat{v}_i}{\partial \widehat{x}_k} \frac{\partial \widehat{w}_j}{\partial \widehat{x}_\ell}\, d\widehat{x}.$$

The uniform convergence is ensured due to the inequalities (39) and (40) and to the compactness of the domain $\overline{\Omega}_0^F$.

The elementary rules for computing Fréchet derivative establish the identity (41), which completes the proof. □

Lemma 5 *The function*

$$u \in \mathcal{U}_{ad} \mapsto \widehat{f}^F\left(u\right) \in \widehat{W}'$$

is Fréchet differentiable and the derivative $D\widehat{f}^F\left(u\right) \in \mathcal{L}\left(U, \widehat{W}'\right)$ *has the form*

$$\left\langle D\widehat{f}^F\left(u\right)\psi, \widehat{w}\right\rangle = \sum_{i=1}^{2} \int_{\Omega_0^F} \frac{\psi\left(\widehat{x}_1\right)}{H} f_i^F \widehat{w}_i\, d\widehat{x}, \quad \forall \widehat{w} \in \widehat{W}$$

Proof. The above function is affine and from the Proposition 7, it is continuous, then it is Fréchet differentiable. □

Lemma 6 *We have*

a) *the function from* $\mathcal{U}_{ad} \times \widehat{M}$ *into* $\widehat{W} \times \widehat{Q}$ *defined by*

$$\left(u, \widehat{\lambda}\right) \longmapsto \left(\widehat{v}\left(u, \widehat{\lambda}\right), \widehat{p}\left(u, \widehat{\lambda}\right)\right)$$

is Fréchet differentiable on $\mathcal{U}_{ad} \times \widehat{M}$,

b) *the derivative of the function*

$$\widehat{\lambda} \in \widehat{K} \longmapsto \widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \in \widehat{W}$$

has the form

$$-\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) A_S^{-1} C_S + \frac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right).$$

Proof. a) Let $\widehat{\lambda}$ be in $\mathcal{F}_{ad}$. We have that $u\left(\widehat{\lambda}\right)$ computed from (43) belongs to $\mathcal{U}_{ad}$.

Let $\widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)$ and $\widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)$ be computed from (44).

We recall that $\mathcal{U}$ defined by (35) is an open set in U.

We apply the result concerning the differentiability of the implicit function (see the Theorem 7 in the Appendix) in the case

$$X = \widehat{M} \times U,\ Y = Z = \widehat{W} \times \widehat{Q},\ G = \widehat{M} \times \mathcal{U} \times \widehat{W} \times \widehat{Q},$$
$$x_0 = \left(\widehat{\lambda}, u\left(\widehat{\lambda}\right)\right),\ y_0 = \left(\widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right), \widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)\right),$$
$$x = (\widehat{\mu}, u),\ y = (\widehat{w}, \widehat{q})$$

for the function $h : \left(\widehat{M} \times \mathcal{U}\right) \times \left(\widehat{W} \times \widehat{Q}\right) \to \widehat{W}' \times \widehat{Q}'$ defined by

$$h\left((\widehat{\mu}, u), (\widehat{w}, \widehat{q})\right) = \left(A_F(u)\widehat{w} + B_F^*(u)\widehat{q} - \widehat{f}^F(u) - C_F\widehat{\mu},\ B_F(u)\widehat{w}\right).$$

In Section 7, we have proved that all the hypotheses of the Theorem 4 hold for the previous choice. It remains to show that $\frac{\partial h}{\partial x}$ exists on G and it is continuous in (x_0, y_0).

In order to prove that, we apply the Theorem 2 (see the Appendix). We have to prove that the functions $\frac{\partial h}{\partial \widehat{\mu}}$ and $\frac{\partial h}{\partial u}$ exist on G and they are continuous in (x_0, y_0).

But the function

$$\widehat{\mu} \in \widehat{M} \longmapsto h\left((\widehat{\mu}, u), (\widehat{w}, \widehat{q})\right)$$

is linear and continuous. Its Fréchet derivative is

$$\frac{\partial h}{\partial \widehat{\mu}}\left((\widehat{\mu}, u), (\widehat{w}, \widehat{q})\right) = (-C_F, 0),$$

which is evidently continuous on G (because it is constant).

Next, we prove the similar result for $\frac{\partial h}{\partial u}$.

We obtain from the identities (41) and (42) that there exist three operators

$$\begin{aligned} DA_F(u) &\in \mathcal{L}\left(\widehat{W}, \mathcal{L}\left(U, \widehat{W}'\right)\right) \\ DB_F^*(u) &\in \mathcal{L}\left(\widehat{Q}, \mathcal{L}\left(U, \widehat{W}'\right)\right) \\ DB_F(u) &\in \mathcal{L}\left(\widehat{W}, \mathcal{L}\left(U, \widehat{Q}'\right)\right) \end{aligned}$$

such that

$$\begin{aligned} ((DA_F(u)\,\widehat{v})\,\psi)\,\widehat{w} &= \frac{\partial \widehat{a}_F}{\partial u}(u, \widehat{v}, \widehat{w})\,\psi \\ ((DB_F^*(u)\,\widehat{q})\,\psi)\,\widehat{w} &= \frac{\partial \widehat{b}_F}{\partial u}(u, \widehat{w}, \widehat{q})\,\psi \\ ((DB_F(u)\,\widehat{w})\,\psi)\,\widehat{q} &= \frac{\partial \widehat{b}_F}{\partial u}(u, \widehat{w}, \widehat{q})\,\psi \end{aligned}$$

for all $u \in \mathcal{U}$, $\widehat{v}, \widehat{w} \in \widehat{W}$, $\widehat{q} \in \widehat{Q}$ and $\psi \in U$.

From the Lemma 4, we get that there exists a function ω, such that

$$\widehat{a}_F(\overline{u} + \psi, \widehat{v}, \widehat{w}) - \widehat{a}_F(\overline{u}, \widehat{v}, \widehat{w}) - \frac{\partial \widehat{a}_F}{\partial u}(\overline{u}, \widehat{v}, \widehat{w})\,\psi = \|\psi\|_U\,\omega(\overline{u}, \widehat{v}, \widehat{w}, \psi)$$

or equivalently

$$\begin{aligned} &\langle A_F(\overline{u} + \psi)\,\widehat{v}, \widehat{w}\rangle_{\widehat{W}', \widehat{W}} - \langle A_F(\overline{u})\,\widehat{v}, \widehat{w}\rangle_{\widehat{W}', \widehat{W}} - \langle (DA_F(\overline{u})\,\widehat{v})\,\psi, \widehat{w}\rangle_{\widehat{W}', \widehat{W}} \\ &= \|\psi\|_U\,\omega(\overline{u}, \widehat{v}, \widehat{w}, \psi) \end{aligned}$$

and

$$\lim_{\psi \to 0} \omega(\overline{u}, \widehat{v}, \widehat{w}, \psi) = 0.$$

In fact, we have that ω converges to 0 uniformly with respect to $\|\widehat{w}\|_{\widehat{W}} \leq 1$. More precisely, we have: $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0, \forall \|\widehat{w}\|_{\widehat{W}} \leq 1, \forall \|u - \overline{u}\|_U \leq \delta_\varepsilon$,

$$|\omega(u, \widehat{v}, \widehat{w}, \psi) - \omega(\overline{u}, \widehat{v}, \widehat{w}, \psi)| \leq \varepsilon.$$

Then the function

$$u \longmapsto A_F(u)\,\widehat{v} \in \widehat{W}'$$

is Fréchet differentiable and its derivative is

$$DA_F(u)\,\widehat{v} \in \mathcal{L}\left(U, \widehat{W}'\right).$$

In a similar way, we obtain that the function

$$u \longmapsto h((\widehat{\mu}, u), (\widehat{w}, \widehat{q})) \in \widehat{W}' \times \widehat{Q}'$$

is Fréchet differentiable and its derivative is

$$\frac{\partial h}{\partial u}((\widehat{\mu}, u), (\widehat{w}, \widehat{q})) = \left(DA_F(u)\,\widehat{w} + DB_F^*(u)\,\widehat{q} - D\widehat{f}^F(u)\,,\ DB_F(u)\,\widehat{w}\right).$$

Following an analogous argument as in the Proposition 6 and Corollary 2, we get that the function $\frac{\partial h}{\partial u}$ is continuous on G.

Now, we can apply the Theorem 7 (see the Appendix) and we obtain that the implicit function $\theta : \widehat{M} \times \mathcal{U} \longrightarrow \widehat{W} \times \widehat{Q}$, given by

$$\theta\left(\widehat{\mu}, u\right) = \left(\widehat{v}\left(u, \widehat{\mu}\right), \widehat{p}\left(u, \widehat{\mu}\right)\right),$$

is Fréchet differentiable, which states the first part of this Lemma.

b) Next, from the identity $A_S\, u\left(\widehat{\lambda}\right) = -C_S\,\widehat{\lambda} + f^S$, we have that the function $\widehat{\lambda} \mapsto u\left(\widehat{\lambda}\right)$ is differentiable and $u'\left(\widehat{\lambda}\right) = -A_S^{-1}\, C_S$.

By using the chain rule, the derivative of the function

$$\widehat{\lambda} \mapsto \widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)$$

has the form

$$\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) u'\left(\widehat{\lambda}\right) + \frac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) = -\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) A_S^{-1} C_S + \frac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right)$$

and the proof is complete. □

Now, we present the main result concerning the computation of the gradient for the fluid-structure interaction problem.

Theorem 1 *The cost function j defined by (32) is Fréchet differentiable. Moreover, we have forall $\widehat{\lambda}$ in $\mathcal{F}_{ad}$ and forall $\widehat{\mu}$ in $\widehat{M}$:*

$$\begin{aligned} j'\left(\widehat{\lambda}\right)\widehat{\mu} \quad = \quad & \left(\frac{\partial \widehat{a}_F}{\partial u}\left(u, \widehat{v}, \widehat{z}\right) + \frac{\partial \widehat{b}_F}{\partial u}\left(u, \widehat{z}, \widehat{p}\right) + \frac{\partial \widehat{b}_F}{\partial u}\left(u, \widehat{v}, \widehat{r}\right)\right) A_S^{-1} C_S \widehat{\mu} \\ & - \int_{\Omega_0^F} \frac{\left(A_S^{-1} C_S \widehat{\mu}\right)\left(\widehat{x}_1\right)}{H} f^F \cdot \widehat{z}\, d\widehat{x} + \int_{\Gamma_0} \widehat{v}_2 \frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}}\left(u, \widehat{\lambda}\right) \widehat{\mu}\, d\widehat{\sigma}, \end{aligned}$$

where the displacement u is computed from

$$A_S\, u = -C_S\, \widehat{\lambda} + f^S, \tag{43}$$

the velocity $\widehat{v}$ and the pressure $\widehat{p}$ of the fluid are computed as solution of

$$\left\{ \begin{array}{lcl} \widehat{a}_F\left(u, \widehat{v}, \widehat{w}\right) + \widehat{b}_F\left(u, \widehat{w}, \widehat{p}\right) & = & \left\langle \widehat{f}^F\left(u\right), \widehat{w}\right\rangle + \int_{\Gamma_0} \widehat{\lambda}\widehat{w}_2\, d\widehat{\sigma}, \ \forall \widehat{w} \in \widehat{W} \\ \widehat{b}_F\left(u, \widehat{v}, \widehat{q}\right) & = & 0, \quad \forall \widehat{q} \in \widehat{Q}, \end{array} \right. \tag{44}$$

the adjoint state $\widehat{z}$ and $\widehat{r}$ are computed as solution of

$$\left\{ \begin{array}{lcl} \widehat{a}_F\left(u, \widehat{w}, \widehat{z}\right) + \widehat{b}_F\left(u, \widehat{w}, \widehat{r}\right) & = & \int_{\Gamma_0} \widehat{v}_2 \widehat{w}_2\, d\widehat{\sigma}, \quad \forall \widehat{w} \in \widehat{W} \\ \widehat{b}_F\left(u, \widehat{z}, \widehat{q}\right) & = & 0, \quad \forall \widehat{q} \in \widehat{Q} \end{array} \right. \tag{45}$$

and $\frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu}$ *is computed from*

$$\begin{cases} \widehat{a}_F\left(u,\dfrac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu},\widehat{w}\right)+\widehat{b}_F\left(u,\widehat{w},\dfrac{\partial \widehat{p}}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu}\right) & = \displaystyle\int_{\Gamma_0}\widehat{\mu}\widehat{w}_2\,d\widehat{\sigma},\forall \widehat{w}\in\widehat{W} \\ \widehat{b}_F\left(u,\dfrac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu},\widehat{q}\right) & = 0,\forall \widehat{q}\in\widehat{Q}. \end{cases} \tag{46}$$

Proof. According to the Lemma 3, Lemma 6 and the chain rule, we obtain that j is differentiable and

$$\begin{aligned} j'\left(\widehat{\lambda}\right)\widehat{\mu} &= J'\left(\widehat{v}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\right)\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu} \\ &\quad +J'\left(\widehat{v}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\right)\frac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\widehat{\mu} \\ &= \int_{\Gamma_0}\widehat{v}_2\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\frac{\partial \widehat{v}_2}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu}\,d\widehat{\sigma} \\ &\quad +\int_{\Gamma_0}\widehat{v}_2\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\widehat{\mu}\,d\widehat{\sigma}. \end{aligned}$$

Our next objective is to evaluate the first term of the above sum.

For this, let $(\widehat{z},\widehat{r})$ be the solution of the adjoint system (45). Next, replacing $\widehat{w}$ by $\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu}$ in (45), we obtain:

$$\begin{aligned} &\int_{\Gamma_0}\widehat{v}_2\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\frac{\partial \widehat{v}_2}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu}\,d\widehat{\sigma} \\ &=\widehat{a}_F\left(u,\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu},\widehat{z}\right)+\widehat{b}_F\left(u,\frac{\partial \widehat{v}}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu},\widehat{r}\right). \end{aligned}$$

Next, if we derive the equations (27) with respect to u, we obtain

$$\begin{aligned} &\frac{\partial \widehat{a}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{w}\right)\psi+\frac{\partial \widehat{a}_F}{\partial \widehat{v}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{w}\right)\frac{\partial \widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)\psi \\ &+\frac{\partial \widehat{b}_F}{\partial u}\left(u,\widehat{w},\widehat{p}\left(u,\widehat{\lambda}\right)\right)\psi+\frac{\partial \widehat{b}_F}{\partial \widehat{q}}\left(u,\widehat{w},\widehat{p}\left(u,\widehat{\lambda}\right)\right)\frac{\partial \widehat{p}}{\partial u}\left(u,\widehat{\lambda}\right)\psi \\ &=\int_{\Omega_0^F}\frac{\psi\left(\widehat{x}_1\right)}{H}f^F\cdot\widehat{w}\,d\widehat{x},\quad \forall \widehat{w}\in\widehat{W},\ \forall\psi\in U \end{aligned} \tag{47}$$

and $\forall \widehat{q}\in\widehat{Q}$, $\forall\psi\in U$ we have

$$\frac{\partial \widehat{b}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{q}\right)\psi+\frac{\partial \widehat{b}_F}{\partial \widehat{w}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{q}\right)\frac{\partial \widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)\psi=0. \tag{48}$$

Now, replacing $\widehat{w}$ by $\widehat{z}$ in (47), $\widehat{q}$ by $\widehat{r}$ in (48) and ψ by $A_S^{-1}C_S\widehat{\mu}$ in (47) and (48), we obtain

$$\frac{\partial \widehat{a}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{z}\right)A_S^{-1}C_S\widehat{\mu}+\frac{\partial \widehat{a}_F}{\partial \widehat{v}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{z}\right)\frac{\partial \widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu}$$

$$+\frac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{z},\widehat{p}\left(u,\widehat{\lambda}\right)\right)A_S^{-1}\,C_S\widehat{\mu}+\frac{\partial\widehat{b}_F}{\partial\widehat{q}}\left(u,\widehat{z},\widehat{p}\left(u,\widehat{\lambda}\right)\right)\frac{\partial\widehat{p}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}\,C_S\widehat{\mu}$$
$$=\int_{\Omega_0^F}\frac{\left(A_S^{-1}C_S\widehat{\mu}\right)\left(\widehat{x}_1\right)}{H}f^F\cdot\widehat{z}\,d\widehat{x} \tag{49}$$

and

$$\tfrac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{r}\right)A_S^{-1}C_S\widehat{\mu}+\tfrac{\partial\widehat{b}_F}{\partial\widehat{w}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{r}\right)\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu}=0. \tag{50}$$

But the following functions

$$\widehat{v}\mapsto\widehat{a}_F\left(u,\widehat{v},\widehat{z}\right),\quad \widehat{q}\mapsto\widehat{b}_F\left(u,\widehat{v},\widehat{q}\right),\quad \widehat{w}\mapsto\widehat{b}_F\left(u,\widehat{w},\widehat{r}\right)$$

are linear and continuous. Consequently, they are differentiable and we have

$$\begin{aligned}
&\tfrac{\partial\widehat{a}_F}{\partial\widehat{v}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{z}\right)\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu}=\widehat{a}_F\left(u,\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu},\widehat{z}\right)\\
&\tfrac{\partial\widehat{b}_F}{\partial\widehat{q}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{z}\right)\tfrac{\partial\widehat{p}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu}=\widehat{b}_F\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\tfrac{\partial\widehat{p}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu}\right)\\
&\tfrac{\partial\widehat{b}_F}{\partial\widehat{w}}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{r}\right)\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu}=\widehat{b}_F\left(u,\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu},\widehat{r}\right).
\end{aligned} \tag{51}$$

So, the identity (49) could be rewritten as follows

$$\begin{aligned}
&\frac{\partial\widehat{a}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{z}\right)A_S^{-1}\,C_S\widehat{\mu}+\widehat{a}_F\left(u,\frac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}\,C_S\widehat{\mu},\widehat{z}\right)\\
&+\frac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{z},\widehat{p}\left(u,\widehat{\lambda}\right)\right)A_S^{-1}\,C_S\widehat{\mu}+\widehat{b}_F\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\frac{\partial\widehat{p}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}\,C_S\widehat{\mu}\right)\\
&=\int_{\Omega_0^F}\frac{\left(A_S^{-1}C_S\widehat{\mu}\right)\left(\widehat{x}_1\right)}{H}f^F\cdot\widehat{z}\,d\widehat{x}.
\end{aligned}$$

Since $\widehat{b}_F\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{q}\right)=0$ for all $\widehat{q}$, it follows that

$$\begin{aligned}
&\tfrac{\partial\widehat{a}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{z}\right)A_S^{-1}C_S\widehat{\mu}+\tfrac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{z},\widehat{p}\left(u,\widehat{\lambda}\right)\right)A_S^{-1}C_S\widehat{\mu}\\
&-\int_{\Omega_0^F}\tfrac{\left(A_S^{-1}C_S\widehat{\mu}\right)\left(\widehat{x}_1\right)}{H}f^F\cdot\widehat{z}\,d\widehat{x}=-\widehat{a}_F\left(u,\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}C_S\widehat{\mu},\widehat{z}\right)\\
&=\widehat{a}_F\left(u,\tfrac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu},\widehat{z}\right).
\end{aligned}$$

Now, replacing the third equality of (51) in (50), we get

$$\begin{aligned}
\frac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{v}\left(u,\widehat{\lambda}\right),\widehat{r}\right)A_S^{-1}\,C_S\widehat{\mu}&=-\widehat{b}_F\left(u,\frac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)A_S^{-1}\,C_S\widehat{\mu},\widehat{r}\right)\\
&=\widehat{b}_F\left(u,\frac{\partial\widehat{v}}{\partial u}\left(u,\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu},\widehat{r}\right)
\end{aligned}$$

which completes the computation of the first term of the gradient, i.e.

$$\begin{aligned}
&\int_{\Gamma_0}\widehat{v}_2\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)\tfrac{\partial\widehat{v}_2}{\partial u}\left(u\left(\widehat{\lambda}\right),\widehat{\lambda}\right)u'\left(\widehat{\lambda}\right)\widehat{\mu}\,d\widehat{\sigma}\\
&=\left(\tfrac{\partial\widehat{a}_F}{\partial u}\left(u,\widehat{v},\widehat{z}\right)+\tfrac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{z},\widehat{p}\right)+\tfrac{\partial\widehat{b}_F}{\partial u}\left(u,\widehat{v},\widehat{r}\right)\right)A_S^{-1}C_S\widehat{\mu}\\
&\quad-\int_{\Omega_0^F}\tfrac{\left(A_S^{-1}C_S\widehat{\mu}\right)\left(\widehat{x}_1\right)}{H}f^F\cdot\widehat{z}\,d\widehat{x}.
\end{aligned}$$

Our next goal is to compute the second term of the gradient.

The function

$$\widehat{\lambda} \mapsto \begin{pmatrix} \widehat{v}\left(u,\widehat{\lambda}\right) \\ \widehat{p}\left(u,\widehat{\lambda}\right) \end{pmatrix} = \begin{pmatrix} A_F(u) & B_F^*(u) \\ B_F(u) & 0 \end{pmatrix}^{-1} \begin{pmatrix} C_F\widehat{\lambda} \\ 0 \end{pmatrix}$$

is linear and continuous, therefore it is differentiable. Moreover, we have

$$\begin{pmatrix} \frac{\partial \widehat{v}}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu} \\ \frac{\partial \widehat{p}}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu} \end{pmatrix} = \begin{pmatrix} A_F(u) & B_F^*(u) \\ B_F(u) & 0 \end{pmatrix}^{-1} \begin{pmatrix} C_F\widehat{\mu} \\ 0 \end{pmatrix}.$$

So, we can compute $\frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu}$ by solving a Stokes problem which permits to compute numerically the second term of the gradient and the proof is complete. □

9 Approximation and numerical results

In this Section we present a practical application of the optimal control algorithm presented in Section 6, having in view the computation of cost function gradient. For this, we propose particular numerical approximation methods.

Let $\phi_i \in L^2(0,L)$ be some particular given functions and let $\alpha_i \in \mathbb{R}$ be the discret controls to be identified, $1 \le i \le m$.

From the Proposition 2, $ii)$ there exist $u_0 \in U$ such that $\int_0^L u_0\, dx_1 = 0$ and $c_0 \in \mathbb{R}$ solutions of (13) and $u_i \in U$ such that $\int_0^L u_i\, dx_1 = 0$ and $c_i \in \mathbb{R}$ solutions of

$$a_S(u_i,\psi) = \int_0^L (\phi_i(x_1) + c_i)\,\psi(x_1)\, dx_1 \quad \forall \psi \in U. \tag{52}$$

It was not necessary to have $\int_0^L \phi_i(x_1)\, dx_1 = 0$.

We take $\widehat{\lambda}(x_1,H) = -c_0 + \sum_{i=1}^m \alpha_i\left(-\phi_i(x_1) - c_i\right)$ in the equation (43) and we obtain the displacement $u = u_0 + \sum_{i=1}^m \alpha_i u_i$ such that $\int_0^L u\, dx_1 = 0$. In other words, $\widehat{\lambda}(x_1,H) = -c_0 + \sum_{i=1}^m \alpha_i\left(-\phi_i(x_1) - c_i\right)$ is an admissible control if and only if the displacement $u = u_0 + \sum_{i=1}^m \alpha_i u_i$ verifies the condition (4).

With the notation

$$\mathcal{J}(\alpha_1,\ldots,\alpha_m) = j\left(-c_0 + \sum_{i=1}^m \alpha_i\left(-\phi_i(x_1) - c_i\right)\right)$$

we have

$$\begin{aligned} \frac{\partial \mathcal{J}}{\partial \alpha_k}(\alpha_1,\ldots,\alpha_m) &= j'\left(\widehat{\lambda} = -c_0 + \sum_{i=1}^m \alpha_i\left(-\phi_i(\widehat{x}_1) - c_i\right)\right)(-\phi_k - c_k) \\ &= -\frac{\partial \widehat{a}_F}{\partial u}(u,\widehat{v},\widehat{z})\,u_k - \frac{\partial \widehat{b}_F}{\partial u}(u,\widehat{z},\widehat{p})\,u_k - \frac{\partial \widehat{b}_F}{\partial u}(u,\widehat{v},\widehat{r})\,u_k \end{aligned}$$

$$+\int_{\Omega_0^F} \frac{u_k\left(\widehat{x}_1\right)}{H} f^F \cdot \widehat{z}\, d\widehat{x} + \int_{\Gamma_0} \widehat{v}_2 \frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}} \left(u, \widehat{\lambda}\right) \left(-\phi_k - c_k\right) d\widehat{\sigma},$$

where $\widehat{v}$ and $\widehat{p}$ is the solution of (44), $\widehat{z}$ and $\widehat{r}$ is the solution of (45) and $\frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}} \left(u, \widehat{\lambda}\right) \left(-\phi_k - c_k\right)$ is the solution of (46) for $\widehat{\mu} = -\phi_k - c_k$.

The problems (44), (45) and (46) represent weak forms of different Stokes equations written in the reference domain Ω_0^F. We know that (44), for example, is equivalent to (18) which represents a weak form of a Stokes equation written in the real domain Ω_u^F. For the approximation by Finite Elements Method, it is better to use (18) instead of (44), because there exists a large literature concerning mixed form of Stokes equations, see for example the standard works [19] and [6].

The function $\mathcal{J}$ is not defined in whole $\mathbb{R}^m$, but only for $\alpha = (\alpha_1, \ldots, \alpha_m)$ such that the displacement $u = u_0 + \sum_{i=1}^m \alpha_i u_i$ verifies the condition (4). If we ignore for the moment this constraint, so that we can use quasi-Newton methods like Broyden, Fletcher, Goldfarb, Shanno (BFGS) or Davidon, Fletcher, Powell (DFP) algorithms for the minimization problem without constraints

$$\inf \mathcal{J}\left(\alpha_1, \ldots, \alpha_m\right).$$

Constrained minimization algorithms like projected or penalization techniques can also be used.

Among the wide variety of possible applications of the here presented control approach of fluid-structure problems, we are interested in simulating the blood flow through medium vessels (arteries). The computation has been made in a domain of length $L = 3\ cm$ and height $H = 0.5\ cm$ which represents a half width of the vessel. In this case, the fluid is the blood and the structure is the wall of the vessel.

The numerical values of the following physical parameters have been taken from [17]. The viscosity of the blood was taken to be $\mu = 0.035\ \frac{g}{cm \cdot s}$, its density $\rho^F = 1\ \frac{g}{cm^3}$. The thickness of the vessel is $h = 0.1\ cm$, the Young modulus $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the density $\rho^S = 1.1\ \frac{g}{cm^3}$.

The gravitational acceleration is $g_0 = 981\ \frac{cm}{s^2}$ and the averaged volume force of the structure is $f^S(x_1) = -g_0 \rho^S h$.

On the rigid boundary, we impose the following boundary conditions:

$$\begin{aligned} v_1(x_1, x_2) &= \begin{cases} \left(1 - \frac{x_2^2}{H^2}\right) V_0, & (x_1, x_2) \in \Sigma_1 \cup \Sigma_3 \\ V_0, & (x_1, x_2) \in \Sigma_2 \end{cases} \\ v_2(x_1, x_2) &= 0, \quad (x_1, x_2) \in \Sigma \end{aligned}$$

where $V_0 = 30\ \frac{cm}{s}$.[39] The volume force in fluid is $\mathbf{f}^F = (0, -g_0 \rho^F)^T$. Imposing non-homogeneous Dirichlet boundary conditions for the velocity on the rigid boundary do not change the formula to compute the gradient of the cost function, excepting the space where we search the velocity $\widehat{v}$ in the problem (44).

Using the notations from the beginning of this section, we have $c_0 = g_0 \rho^S h$ and $u_0 = 0$.

We take $m = 4$. Let $\xi_i = (i-1)L/(m-1)$ for $1 \leq i \leq m$. There exist ϕ_i polynomial functions of degree 3, such that $\phi_i(\xi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker's symbol.

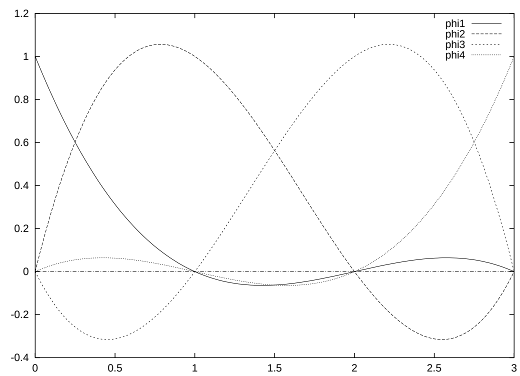

Figure 2: The shape functions ϕ_i for the approximation of the control

Let u_i, c_i be the solutions of (52). From the regularity of ϕ_i, we can use the following strong formulation in order to compute u_i, c_i:

$$\begin{aligned} EI\, u_i''''(x_1) &= \phi_i(x_1) + c_i, \ \forall x_1 \in (0, L) \\ u_i(0) &= u_i'(0) = 0, \\ u_i(L) &= u_i'(L) = 0 \\ \int_0^L u_i(x_1)\, dx_1 &= 0. \end{aligned}$$

We have computed u_i, c_i exactly, using the software *Mathematica*. The displacements u_i are polynomial functions of degree 7. We could use alternatively finite elements shape functions for ϕ_i, but in this case we should handle the weak formulation in order to compute u_i and c_i.

For the fluid we have used a Mixed Finite Elements Method, $P2$ Lagrange triangles for the velocity and $P1$ for the pressure [19], [6].

The numerical tests have been produced using *freefem++ v1.27*.[24] We have used the BFGS algorithm for the minimization problem with the starting point $\alpha = 0$ so that in the first five iterations the cost function takes the values presented in Table 1.

After 5 iterations we have obtained

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (13.81347223,\ 2.81316723,\ -2.64008687,\ -13.98655258)$$

and the gradient of the cost function was

$$\nabla \mathcal{J} = (0.000255,\ 0.004768,\ -0.020800,\ 0.009256)^T.$$

More iterations do not quantitatively change the values of α, the cost function and the solution. The relative change in successive values of α evaluated in the

Iterations	$\mathcal{J}$
0	8.369704278
1	7.705856075
2	0.152977642
3	0.147957298
4	0.145206068
5	0.144683623

Table 1: The cost function history

norm $\|\cdot\|_\infty$ is less than 0.02. The first four digits to the right of the decimal point of the cost function don't change after the fifth iteration. Ten iterations are required to achieve $\|\nabla\mathcal{J}\|_\infty \leq 10^{-6}$.

Notice that the condition (4) was not violated.

In order to compute $\nabla\mathcal{J}(\alpha)$, we have to solve the adjoint state problem (45) and m linear systems (46) which have the same matrix. The linear systems were solved by LU decomposition. We observe that (44) and (45) have the same left side, so when we compute $\nabla\mathcal{J}(\alpha)$ we can use the same LU decomposition obtained computing $\mathcal{J}(\alpha)$. If we compute $\nabla\mathcal{J}(\alpha)$ by the Finite Differences Method, we have to solve m linear systems, but the matrices are different because u changes, so using the analytic formula of the gradient is more advantageous.

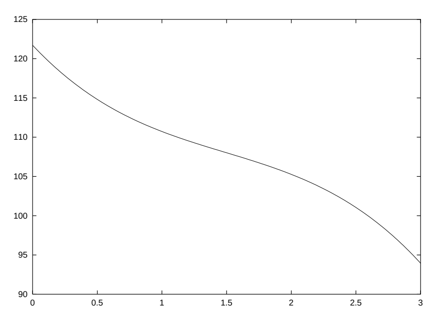

Figure 3: The applied stresses $-\widehat{\lambda} = c_0 + \sum_{i=1}^{m} \alpha_i \left(\phi_i + c_i\right)$ $[dyn/cm^2]$ on the interface

We have obtained $\left\|\widehat{\lambda} + \widehat{p}\right\|_{0,\Gamma_0}^2 = 0.002878$, in other words, the vertical component of the stresses exerced by the fluid on the interface depends on the pressure only, $-\widehat{\lambda} \approx \widehat{p}_{|\Gamma_0}$. This is justified by the following result [34]: if $\mathbf{v} \in \left(H^2\left(\Omega_u^F\right)\right)^2$, $p \in H^1\left(\Omega_u^F\right)$, $\mathbf{v}$ is constant on Γ_u, div $\mathbf{v} = 0$ in Ω_u^F, then $-\left(\sigma^F\mathbf{n}\right)\cdot\mathbf{n} = p$ on Γ_u. In our case $-\left(\sigma^F\mathbf{n}\right)\cdot e_2 = -\lambda$ and $\mathbf{n} \approx e_2$.

As we see in Figure 4, the velocity on the boundary Γ_0 is not null, but the

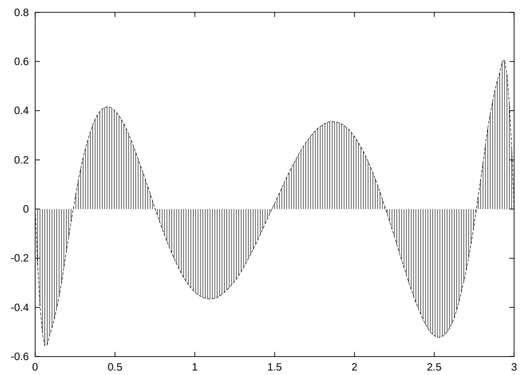

Figure 4: The velocity $[cm/s]$ on the boundary Γ_0

maximum of the absolute value is less than 0.6 cm/s.

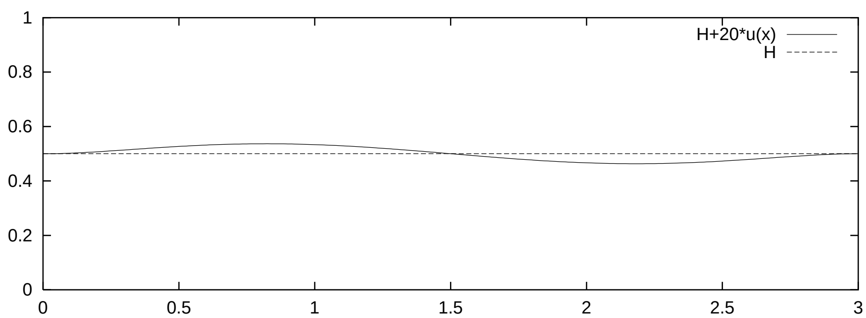

Figure 5: The displacement $[cm]$ of the vessel magnified by a factor 20

The displacement of the vessel is very small, it is less than 0.04 cm (see Figure 5). The pressure on the interface $\widehat{p}$ is almost the same as $-\widehat{\lambda}$, so it decreases from the inflow (left) to the outflow (right). The displacement of the interface is consequent with the pressure: the displacement of the vessel wall is outwards at the left and inwards at the right.

The computed velocity distribution is similar to a Poiseuille flow (see Figure 6).

10 Conclusions

In this work, a particular fluid-structure interaction model is formulated as an optimal control problem.

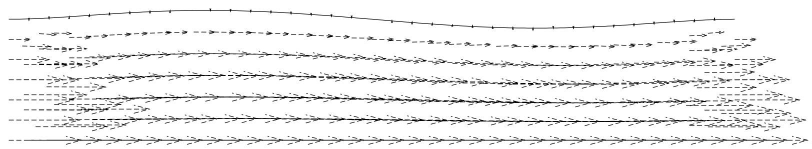

Figure 6: The velocity $[cm/s]$ reduced by a factor 100

The optimal control setting allows to solve numerically the fluid-structure interaction problem (which is *a priori* a coupled problem) by an iterative algorithm such that the fluid and the structure equations are solved separately at each iteration. Thus, existing software packages could be adapted to approximate the solution of all the intermediate problems appearing in the algorithm.

The differentiability of the cost function and the analytical expression for its gradient are obtained.

In order to perform a numerical method, the analytic expression for the gradient reveals very useful and accurate to apply classical descent methods. To solve numerically a problem arising from blood flow in arteries, we have used a quasi Newton method which employs the analytic gradient of the cost function and the approximation of the inverse Hessian is updated by the Broyden, Fletcher, Goldforb, Shano (BFGS) scheme. This algorithm is faster than fixed point with relaxation or block Newton methods.

We can adapt this technique to the unsteady coupled fluid-structure problems.

A Appendices

A.1 Fréchet differentiability

Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be three normed spaces.

Definition 1 *We say that the function $f : X \to Y$ is **Fréchet differentiable at** $x \in X$, if there exists $f'(x) \in \mathcal{L}(X, Y)$ such that*

$$\lim_{h \to 0} \frac{\|f(x+h) - f(x) - f'(x)\, h\|_Y}{\|h\|_X} = 0$$

*The linear operator $f'(x)$ is called the **Fréchet derivative of** f **at** x.*

In the case when $X = \prod_{i=1}^{n} X_i$, we denote by $\frac{\partial f}{\partial x_i}(x) \in \mathcal{L}(X_i, Y)$ the **Fréchet partial derivative of** f **with respect to** x_i **at** $x \in X$.

Theorem 2 *Let $f : X = \prod_{i=1}^{n} X_i \to Y$ be a function and let x^0 be an element of X. We assume that there exists V a neighborhood of x^0, such that $\frac{\partial f}{\partial x_i}$ exists on V and its are continuous in x^0.*

Then f is Fréchet differentiable in x^0 and

$$f'\left(x^0\right) h = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(x^0\right) h_i$$

for all $h = (h_1, \ldots, h_n) \in X$.

Theorem 3 *Let $h : X \to Z$ be the composition of two mappings $f : X \to Y$ and $g : Y \to Z$*

$$h = g \circ f$$

Assume that f is Fréchet Differentiable in x and g in $f(x)$, then h is Fréchet differentiable in x and

$$h'(x) = g'(f(x)) \circ f'(x).$$

A.2 Implicit Function Theorem

We begin by recalling the Implicit Function Theorem. The proof of this result could be found in [25] for example.

Theorem 4 (The Implicit Function Theorem) *Let $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ and $(Z, \|\cdot\|_Z)$ be normed spaces. We suppose that h is a mapping from an open subset G of $X \times Y$ into Z.*

Suppose (x_0, y_0) is a point in G and h is continuous in (x_0, y_0) such that:

i) $h(x_0, y_0) = 0$,

ii) $\frac{\partial h}{\partial y}$ *exists on G and it is continuous in (x_0, y_0),*

iii) $\frac{\partial h}{\partial y}(x_0, y_0)$ *is invertible and* $\left(\frac{\partial h}{\partial y}(x_0, y_0)\right)^{-1}$ *is continuous.*

Then there exists a neighborhood V of x_0 and a function $\theta : V \to Z$ such that:

iv) $\theta(x_0) = y_0$,

v) $h(x, \theta(x)) = 0$ *for all x in V,*

vi) θ *is continuous in x_0.*

Remark 5 *If h is continuous on G, then θ is continuous in a neighborhood of x_0.*

Remark 6 *In the case when X, Y and Z are Banach spaces, if $\frac{\partial h}{\partial y}(x_0, y_0) \in \mathcal{L}(Y, Z)$ is invertible, from the Open Mapping Theorem we have that $\left(\frac{\partial h}{\partial y}(x_0, y_0)\right)^{-1}$ is continuous.*

Theorem 5 (The differentiability of the implicit funtion) *Moreover, if there exists $\frac{\partial h}{\partial x}$ on G continuous in (x_0, y_0), then the implicit function θ is Fréchet differentiable in x_0 and*

$$\theta'(x_0) = -\left(\frac{\partial h}{\partial y}(x_0, y_0)\right)^{-1} \frac{\partial h}{\partial x}(x_0, y_0).$$

A.3 Integrals with parameter

Let U be a Hilbert space and let $\widehat{\Omega}$ be a compact set of $\mathbb{R}^2$.

Theorem 6 (continuity of integrals with parameter) *Let f be a function from $U \times \widehat{\Omega}$ to $\mathbb{R}$ such that for all $u \in U$ the function*

$$\widehat{x} \in \widehat{\Omega} \longmapsto f(u, \widehat{x})$$

is Lebesgue integrable.

Let $\overline{u}$ be an element of U such that $f(u, \widehat{x})$ converges to $f(\overline{u}, \widehat{x})$ uniformly with respect to $\widehat{x}$, when u converges to $\overline{u}$.

Then

$$\lim_{u \to \overline{u}} \int_{\widehat{\Omega}} f(u, \widehat{x})\, d\widehat{x} = \int_{\widehat{\Omega}} f(\overline{u}, \widehat{x})\, d\widehat{x}.$$

Theorem 7 (differentiability of integrals with parameter) *Moreover, we assume that:*

a) *for all $u \in U$ and for all $\widehat{x} \in \widehat{\Omega}$, the Fréchet derivative*

$$\frac{\partial f}{\partial u}(u, \widehat{x}) \in U'$$

exists,

b) *the functions*

$$\widehat{x} \in \widehat{\Omega} \longmapsto \frac{\partial f}{\partial u}(u, \widehat{x})\, \psi \in \mathbb{R}$$

are Lebesgue integrable for all $\psi \in U$,

c) *$\frac{\partial f}{\partial u}(u, \widehat{x})$ converges in U' to $\frac{\partial f}{\partial u}(\overline{u}, \widehat{x})$ uniformly with respect to $\widehat{x}$, when u converges to $\overline{u}$.*

Then, the function F from U to $\mathbb{R}$, defined by

$$F(u) = \int_{\widehat{\Omega}} f(u, \widehat{x})\, d\widehat{x}$$

is Fréchet differentiable in $\overline{u}$ *and*

$$F'(\overline{u})\,\psi = \int_{\widehat{\Omega}} \frac{\partial f}{\partial u}(\overline{u},\widehat{x})\,\psi d\widehat{x}$$

for all $\psi \in U$.

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