ELASTIC HELICES

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There are two variational problems for space curves inspired by the mechanics of elastic rods: the first one was formulated by Daniel Bernoulli (1740) for rods with circular cross-section and the second one was formulated by M. Sadowsky (1930) for rectangular narrow thin plates. The helical solutions of the first problem are always circular. Sadowsky's problem admits all the circular helices as solutions and in addition also non circular helices.

These last are geodesics on cylinders having as orthogonal cross section Poleni's curve (1729) called also "la courbe des forçats".

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1. INTRODUCTION

There are two variational problems for space curves inspired by the mechanics of elastic rods. The equilibrium of rods with circular cross section leads to the study of the functional

$$\mathcal{B}(s) = \int_c \kappa^2 ds$$

considered firstly by Daniel Bernoulli (1740) and then by Euler (1743). The equilibrium of narrow thin plates leads to the functional

$$\mathcal{S}(c) = \int_c \kappa^2 \left(1 + \frac{\tau^2}{\kappa^2}\right)^2 ds$$

introduced by Sadowsky in 1930. Here c is a space curve, κ and τ are respectively its curvature and torsion and s denotes an arc-length parameter. Extremals of $\mathcal{B}(c)$, when helices, are circular i.e. they have constant κ and τ. All circular helices are extremals for $\mathcal{S}(c)$ but there are also non-circular helices, i.e. curves c along which only the ratio $\omega = \kappa^{-1}\tau$ is constant, which are extremals for $\mathcal{S}(c)$; these are geodesics on cylinders having as directrix curves special syntractrices the so-called Poleni's curves (1729). The paper contains the proof of this result.

2. EULER-LAGRANGE EQUATIONS

The Euler-Lagrange equations corresponding to the functional $\mathcal{B}(c)$ are

$$\kappa'' - \kappa\tau^2 + \frac{1}{2}\kappa^3 = 0 \tag{1}$$

$$2\kappa'\tau + \kappa\tau' = 0 \tag{2}$$

where primes denote derivation with respect to arc-length s. If $\omega = \kappa^{-1}\tau$ is constant, the equation (2) reduces to $\kappa\kappa^{'}\omega = 0$ so that the product $\kappa^2\omega$ is constant and therefore the curvature κ is also constant; the helices solutions of the system (1)–(2) are therefore circular.

Following [1], the Euler-Lagrange equations corresponding to the functional $\mathcal{S}(c)$ are

$$\begin{aligned}
&\kappa^{-1}(1+\omega^2)\kappa''' + 2\omega\omega''' \\
&+\kappa''(-\kappa^{-2}(1+\omega^2)\kappa' + 12\kappa^{-1}\omega\omega') \\
&+\omega''(6\kappa^{-1}\omega\kappa' + 8(1+\omega^2)^{-1}(1+3\omega^2)\omega') \\
&-8\kappa^{-2}\omega\left(\kappa'\right)^2\omega' + 8\kappa^{-1}(1+\omega^2)^{-1}(1+3\omega^2)\kappa'\left(\omega'\right)^2 \\
&+24\omega(1+\omega^2)^{-1}\left(\omega'\right)^3 + \kappa(1+\omega^2)^2\kappa' + 3\kappa^2\omega(1+\omega^2)\omega' = 0
\end{aligned} \tag{3}$$

and

$$\begin{aligned}
&2\kappa^{-1}\omega(1+\omega^2)\kappa''' + 2(1+3\omega^2)\omega''' \\
&+\kappa''(-6\kappa^{-2}\omega(1+\omega^2)\kappa' + 4\kappa^{-1}(1+3\omega^2)\omega') \\
&+\omega''(2\kappa^{-1}(1+3\omega^2)\kappa' + 36\omega\omega') \\
&+4\kappa^{-3}\omega(1+\omega^2)\left(\kappa'\right)^3 - 4\kappa^{-2}(1+3\omega^2)\left(\kappa'\right)^2\omega' + 12\kappa^{-1}\omega\kappa'\left(\omega'\right)^2 \\
&+12\left(\omega'\right)^3 + \kappa\omega(1+\omega^2)^2\kappa' + \kappa^2(1+\omega^2)(1+3\omega^2)\omega' = 0.
\end{aligned} \tag{4}$$

These are identically satisfied when κ and $\omega = \kappa^{-1}\tau$ are constant. We look now for solutions with variable κ and constant ω. Under this hypothesis, the system reduces, after the elimination of the third order derivative of κ, to the second order equation

$$\kappa^{-1}\kappa'' - \kappa^{-2}\kappa'^2 + \frac{1+\omega^2}{4}\kappa^2 = 0.$$

Replacing κ by R^{-1} where R is the radius of curvature of the extremal c, the equation writes

$$-RR^{''} + R^{'2} + \frac{1+\omega^2}{4} = 0.$$

The general solution of this equation is

$$R = \frac{\sqrt{1+\omega^2}}{2\alpha}\cosh(\alpha s + \beta)$$

which depends on two arbitrary constants $\alpha \neq 0$ and β.

3. HELICES

One knows that helices are geodesics on cylinders. In order to determine the directrix i.e. the orthogonal cross-section of the cylinder on which lie the helical extremals of $\mathcal{S}(c)$, lets denote by $\{T, N, B\}$ the Frenet frame of the helix and look for a curve $\gamma(s)$ orthogonal to the generatrices of the cylinder. At each point $c(s)$ of the helix, the vector of Darboux

$$(\omega T + B)/\sqrt{1+\omega^2}$$

defines the direction of the generatrix of the cylinder at $c(s)$. One verifies that the curve

$$\gamma(s) = c(s) - s\frac{\omega}{1+\omega^2}(\omega T(s) + B(s))$$

which lies on the cylinder described by the line which rests on $c(s)$ preserving the fixed direction of Darboux's vector, is orthogonal to the generatrices of the cylinder.

Precisely

$$\gamma'(s) = \frac{1}{1+\omega^2}(T(s) - \omega B(s)) = \left(1+\omega^2\right)^{-1/2}(T(s) - \omega B(s))\left(1+\omega^2\right)^{-1/2}$$

so that

$$\sigma = s\left(1+\omega^2\right)^{-1/2}$$

is the arc-length on γ. Moreover, as the derivative with respect to σ of the unitary tangent vector to γ, $(T(s) - \omega B(s))\left(1+\omega^2\right)^{-1/2}$ is

$$\left(1+\omega^2\right)^{1/2}\left(\frac{d}{ds}(T(s) - \omega B(s))\left(1+\omega^2\right)^{-1/2}\right) = \frac{1}{R}\left(1+\omega^2\right)N$$

one deduces that the radius of curvature $r(s)$ of the curve γ is

$$r(s) = R\left(1+\omega^2\right)^{-1} = \frac{1}{2\alpha\sqrt{1+\omega^2}}\cosh(\alpha s + \beta).$$

In order to determine now the plane curve γ whose radius of curvature r is known, one has to integrate the differential system

$$\frac{d^2x}{d\sigma^2} = -\frac{1}{r}\frac{dy}{d\sigma}, \quad \frac{d^2y}{d\sigma^2} = \frac{1}{r}\frac{dx}{d\sigma}.$$

One introduces a complex variable $z = x + iy$, so that the above system becomes

$$\frac{d^2z}{d\sigma^2} = \frac{i}{r}\frac{dz}{d\sigma},$$

or

(5)
$$\frac{z''}{z'} = \frac{i2\alpha}{\cosh(\alpha s + \beta)}.$$

A first integration of this equation leads to

$$\begin{aligned}
z' &= e^C e^{4i \arctan e^{\alpha s+\beta}} \\
&= e^C \left(\cos\left(4 \arctan e^{\alpha s+\beta}\right) + i \sin\left(4 \arctan e^{\alpha s+\beta}\right)\right) \\
&= e^C \left(\cos\left(\arctan e^{\alpha s+\beta}\right) + i \sin\left(\arctan e^{\alpha s+\beta}\right)\right)^4 \\
&= e^C \left(1 - \frac{8e^{2(\alpha s+\beta)}}{\left(1+e^{2(\alpha s+\beta)}\right)^2} + i4\frac{e^{\alpha s+\beta}\left(1-e^{2(\alpha s+\beta)}\right)}{\left(1+e^{2(\alpha s+\beta)}\right)^2}\right).
\end{aligned}$$

We have

$$\frac{8}{2\alpha}\left(\frac{1}{1+e^{2(\alpha s+\beta)}}\right)' = \frac{4}{\alpha}\left(-\frac{2\alpha e^{2(\alpha s+\beta)}}{\left(1+e^{2(\alpha s+\beta)}\right)^2}\right) = -\frac{8e^{2(\alpha s+\beta)}}{\left(1+e^{2(\alpha s+\beta)}\right)^2}$$

and

$$\begin{aligned}
\frac{4}{\alpha}\left(\frac{e^{\alpha s+\beta}}{1+e^{2(\alpha s+\beta)}}\right)' &= \frac{4}{\alpha}\left[\alpha\left(\frac{e^{\alpha s+\beta}}{1+e^{2(\alpha s+\beta)}} - \frac{e^{\alpha s+\beta}2e^{2(\alpha s+\beta)}}{\left(1+e^{2(\alpha s+\beta)}\right)^2}\right)\right] \\
&= \frac{4}{\alpha}\alpha\frac{e^{\alpha s+\beta}\left(1-e^{2(\alpha s+\beta)}\right)}{\left(1+e^{2(\alpha s+\beta)}\right)^2} = 4\frac{e^{\alpha s+\beta}\left(1-e^{2(\alpha s+\beta)}\right)}{\left(1+e^{2(\alpha s+\beta)}\right)^2}.
\end{aligned}$$

The system (5) has the solutions of the form

$$z = e^C\left(s + \frac{4}{\alpha}\frac{1}{1+e^{2(\alpha s+\beta)}} + \frac{4i}{\alpha}\frac{e^{\alpha s+\beta}}{1+e^{2(\alpha s+\beta)}}\right) + Const$$

In particular, for $C = 0$, $\alpha = 1$, $\beta = 0$, $Const = 0$ and $z = x + iy$, we obtain

$$x(s) = s + \frac{4}{1+e^{2s}}, \quad y(s) = 4\frac{e^s}{1+e^{2s}} = \frac{2}{\cosh s}.$$

Replacing x by $x + 2$ the parametric equations of the curve $\gamma(s)$ becomes

$$x(s) = s - 2\tanh s, \quad y(s) = \frac{2}{\cosh s}$$

and one recognizes the parametric equations of the syntractrix of Poleni, called also “la courbe des forçats” [2].

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