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Extension theorems related to a fluid-structure interaction problem

by

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Dedicated to professor Teodor Morozan
on the occasion of his 80th birthday

Abstract

The aim of this paper is to prove the existence of an approximate weak solution for a steady fluid-structure interaction problem. A fictitious domain approach with penalization is used. One of the main ingredients is an extension theorem for domains with Lipschitz boundaries. The fluid and structure domains are not necessarily double connected and the structure is not completely surrounded by the fluid. These assumptions are more realistic for some engineering and medical applications.

Key Words: fluid-structure interaction, fictituous domain.
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1 Introduction

The present paper is devoted to the study of the behavior of an elastic structure immersed in an incompressible fluid. We study the case where the fluid does not surround completely the structure. This configuration is encountered in diverse engineering and medical applications: artificial heart valves, hydraulic shock absorber, etc. We use Stokes equations to model the flow motion. The displacement of the structure, under the small deformations assumption, will be modeled by linear elasticity equations. Only the steady case will be studied.

Existence for steady interaction between a fluid and an elastic structure was proved in: Rumpf [31], Grandmont [18], [20], Bayada, Chambat, Cid, Vazquez [2], Flori, Giudicelli [14], Surulescu [32], Galdi, Kyed [16]. In these papers, the fluid equations are reformulated in a reference configuration. Consequently, the coefficients of the fluid problem are non-constant and depend on the structure deformation. For the unsteady case, existence results can be found in Grandmont, Maday [19], Desjardins, Esteban, Grandmont, Le Tallec [12], Beirao da Veiga [3], Grandmont [7], [21], Bociu, Toundykov, Zolésio [4]. In the case where the structure is rigid, existence results are presented in [10], [33]. Generally, in the literature, when the fluid and structure domains are both in 2D, or both in 3D, either the fluid completely surrounds the structure or conversely and the domains are regular.

The aim of this paper is to prove the existence of an approximate weak solution for a steady fluid-structure interaction problem under weaker asumptions. We use as in Halanay, Murea, Tiba [22] and [24] a fictitious domain approach with penalization. One of the main ingredients is an extension theorem for domains with Lipschitz boundaries that applies some results from Chenais [8] and Galdi [15]. In Halanay, Murea, Tiba [22], using a non-linear penalization term, a regularization of the characteristic function and regular domains, the regularity $W^{2,p}$, $p>2$ is obtained for the fluid velocity and for the structure displacement. In Halanay, Murea, Tiba [24], the fluid and the structure domains are doubly connected, the structure is completely surrounded by the fluid and the domains are regular, the existence of a weak solution for the fluid-structure interaction problem is obtained. Numerical results using this framework are presented in Murea, Halanay [25] and Halanay, Murea [23]. In the present paper, weaker hypothesis on the geometry are used: the fluid and the structure domains are not necessary double connected and the structure is not completely surrounded by the fluid, i.e. the intersection of the closures of the fluid-structure interface and of the exterior boundary of the fluid is non-empty. This configuration is more realistic for some engineering and medical applications.

Some of the techniques from this work may be compared with certain fixed domain approaches in shape optimization Neittaanmäki, Tiba [29], Neittaanmäki, Sprekels, Tiba [28], already applied to free boundary problems and variational inequalities originating in elasticity Murea, Tiba [26], [27].

2 Setting for a fluid-structure interaction problem

Let $D\subset\mathbb{R}^2$ be a bounded, connected, open set with Lipschitz boundary $\partial D=\overline{\Sigma}_1\cup\overline{\Sigma}_2$ such that $\Sigma_1\cap\Sigma_2=\emptyset$ and $\overline{\Sigma}_1\cap\overline{\Sigma}_2=\{S_1,S_2\}$, see Figure 1.

Let Ω_0^S be the undeformed structure domain, and suppose that it is a bounded, connected, open set. Its boundary is Lipschitz and admits the decomposition $\partial\Omega_0^S=\overline{\Gamma}_D\cup\overline{\Gamma}_0$, such that $\Gamma_D\cap\Gamma_0=\emptyset$. We denote the intersection points $\overline{\Gamma}_D\cap\overline{\Gamma}_0=\{R_1,R_2\}$. On Γ_D we impose zero displacement for the structure. We assume that $\Omega_0^S\subset D$ and $\overline{\Gamma}_D\subset\Sigma_2$.

Suppose that the structure is elastic and denote by $\mathbf{u}=(u_1,u_2):\Omega_0^S\to\mathbb{R}^2$ its displacement. A particle of the structure whose initial position was the point $\mathbf{X}$ will occupy the position $\mathbf{x}=\varphi(\mathbf{X})=\mathbf{X}+\mathbf{u}(\mathbf{X})$ in the deformed domain $\Omega_{\mathbf{u}}^S=\varphi\left(\Omega_0^S\right)$.

We admit that $\Omega_{\mathbf{u}}^S\subset D$ and the fluid occupies $\Omega_{\mathbf{u}}^F=D\setminus\overline{\Omega}_{\mathbf{u}}^S$. The boundary $\Gamma_{\mathbf{u}}=\overline{\Omega}_{\mathbf{u}}^S\cap\overline{\Omega}_{\mathbf{u}}^F$ represents the fluid-structure interface. The boundary of the deformed structure is $\partial\Omega_{\mathbf{u}}^S=\overline{\Gamma}_D\cup\overline{\Gamma}_{\mathbf{u}}$ and the boundary of the fluid domain admits the decomposition $\partial\Omega_{\mathbf{u}}^F=\overline{\Sigma}_1\cup(\Sigma_2\setminus\Gamma_D)\cup\overline{\Gamma}_{\mathbf{u}}$.

We have that $\overline{\Gamma}_D\cap\overline{\Gamma}_{\mathbf{u}}=\{R_1,R_2\}$. We assume that Γ_D and $\Gamma_{\mathbf{u}}$ *meet transversally*, this means that the tangents in R_1 to Γ_D and $\Gamma_{\mathbf{u}}$ are different and the same propriety holds in R_2. In other words, the angles of $\Gamma_{\mathbf{u}}$ and Γ_D in R_1 and R_2 are not 0 or π. Consequently $\partial\Omega_{\mathbf{u}}^F$ is Lipschitz. The fluid-structure geometrical configuration is represented in Figure 1.

The fluid equations are described using Eulerian coordinates, while for the structure equations, the Lagrangian coordinates are employed. The gradients with respect to the Eulerian coordinates $\mathbf{x}\in\Omega_{\mathbf{u}}^S$ of a scalar field q or a vector field $\mathbf{w}$ are denoted by ∇q, $\nabla\mathbf{w}$. The scalar product of two vectors $\mathbf{v}$ and $\mathbf{w}$ of $\mathbb{R}^2$ is denoted as $\mathbf{v}\cdot\mathbf{w}=\sum_{i=1}^2 v_i\,w_i$. If $\sigma=(\sigma_{ij})_{1\le i,j\le 2}$ and $\tau=(\tau_{ij})_{1\le i,j\le 2}$ are two tensors, we denote $\sigma:\tau=\sum_{i=1}^2\sum_{j=1}^2\sigma_{ij}\tau_{ij}$. The divergence operators with respect to the Eulerian coordinates of a vector field $\mathbf{w}$ and of a tensor σ are denoted by $\nabla\cdot\mathbf{w}$ and $\nabla\cdot\sigma$.

Similarly, when the derivatives are with respect to the Lagrangian coordinates $\mathbf{X}=\varphi^{-1}(\mathbf{x})\in\Omega_0^S$, we use the notations: $\nabla_{\mathbf{X}}\mathbf{u}$, $\nabla_{\mathbf{X}}\cdot\mathbf{u}$, $\nabla_{\mathbf{X}}\cdot\sigma$.

If $\mathbf{A}$ is a nonsingular square matrix, we denote by det $\mathbf{A}$, $\mathbf{A}^{-1}$, $\mathbf{A}^T$ its determinant, the inverse and the transposed matrix, respectively. We write cof $\mathbf{A}=(\det\mathbf{A})\left(\mathbf{A}^{-1}\right)^T$ the

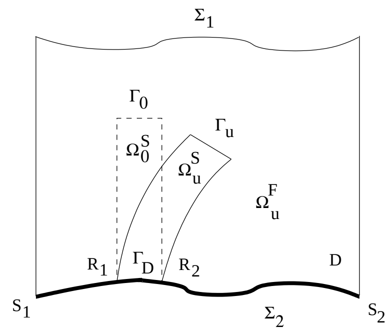

Figure 1: Geometrical configuration. The bold boundary is Σ_2.

co-factor matrix of $\mathbf{A}$. We write $\mathbf{A}^{-T} = \left(\mathbf{A}^{-1}\right)^T$.

We denote by $\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{X})$ the gradient of the deformation and by $J(\mathbf{X}) = \det \mathbf{F}(\mathbf{X})$ the Jacobian determinant, where $\mathbf{I}$ is the unit matrix. We assume that $J(\mathbf{X}) > 0$, for all $\mathbf{X} \in \Omega_0^S$.

Strong formulation

The problem is to find the structure displacement $\mathbf{u} : \overline{\Omega}_0^S \to \mathbb{R}^2$, the fluid velocity $\mathbf{v} : \overline{\Omega}_{\mathbf{u}}^F \to \mathbb{R}^2$ and the fluid pressure $p : \overline{\Omega}_{\mathbf{u}}^F \to \mathbb{R}$ such that:

$$
\begin{aligned}
-\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}) &= \mathbf{f}^S, \quad \text{in } \Omega_0^S && (2.1)\\
\mathbf{u} &= 0, \quad \text{on } \Gamma_D && (2.2)\\
-\nabla \cdot \sigma^F(\mathbf{v}, p) &= \mathbf{f}^F, \quad \text{in } \Omega_{\mathbf{u}}^F && (2.3)\\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_{\mathbf{u}}^F && (2.4)\\
\mathbf{v} &= \mathbf{g}, \quad \text{on } \Sigma_1 && (2.5)\\
\mathbf{v} &= 0, \quad \text{on } \Sigma_2 \setminus \Gamma_D && (2.6)\\
\mathbf{v} &= 0, \quad \text{on } \Gamma_{\mathbf{u}} && (2.7)\\
\omega(\mathbf{X})\, \sigma^F(\mathbf{v}(\mathbf{x}), p(\mathbf{x}))\, \mathbf{n}^F(\mathbf{x}) &= -\sigma^S(\mathbf{u}(\mathbf{X}))\, \mathbf{n}^S(\mathbf{X}), \quad \forall \mathbf{X} \in \Gamma_0,\ \mathbf{x} = \varphi(\mathbf{X}) && (2.8)
\end{aligned}
$$

where $\mathbf{f}^S : \Omega_0^S \to \mathbb{R}^2$ are the applied volume forces on the structure and $\mathbf{n}^S$ is the structure unit outward vector normal to $\partial\Omega_0^S$. Similarly, we define $\mathbf{f}^F : \Omega_{\mathbf{u}}^F \to \mathbb{R}^2$ and $\mathbf{n}^F$ the fluid unit outward vector normal to $\partial\Omega_{\mathbf{u}}^F$. In (2.5), $\mathbf{g} : \Sigma_1 \to \mathbb{R}^2$ is a prescribed velocity, such that $\int_{\Sigma_1} \mathbf{g} \cdot \mathbf{n}^F ds = 0$. Since we look for a continuous solution, the boundary conditions (2.5) and (2.6) must be compatible. We assume that $\mathbf{g} \in \mathcal{C}(\overline{\Sigma_1 \cup \Sigma_2 \setminus \Gamma_D})$ such that $\mathbf{g} = 0$ on $\Sigma_2 \setminus \Gamma_D$.

We have denoted by $\sigma^S(\mathbf{u}) : \Omega_0^S \to \mathbb{R}^4$ the stress tensor of the structure and by $\sigma^F(\mathbf{v}, p) : \Omega_{\mathbf{u}}^F \to \mathbb{R}^4$ the Cauchy stress tensor of the fluid. We point out that the stress tensor of the structure is defined on the undeformed structure domain Ω_0^S and it will be the linear version of the Piola-Kirchoff tensor. The Cauchy stress tensor of the fluid is defined in the deformed domain $\Omega_{\mathbf{u}}^F$. The constitutive relations will be precised later.

We have used the notation $\omega(\mathbf{X}) = \left\| J(\mathbf{X})\ \mathbf{F}^{-T}(\mathbf{X})\ \mathbf{n}^S(\mathbf{X}) \right\|_{\mathbb{R}^2} = \left\| \mathrm{cof}(\mathbf{F}(\mathbf{X}))\, \mathbf{n}^S(\mathbf{X}) \right\|_{\mathbb{R}^2}$ for $\mathbf{X}$ on $\partial\Omega_0^S$, which is a kind of Jacobian determinant for the change of variable formula for integral over surface, see Ciarlet [9], section 1.7. The equation (2.8) represents the action and reaction principle: the forces acting on the fluid-structure interface are equal in size and opposite in direction.

3 Weak formulation using fictitious domain technique with penalization

Denote by $\|\cdot\|_{m,s,\Omega}$ the usual norm of the Sobolev space $W^{m,s}(\Omega)$. When $s=2$, we use the well known notation $H^m(\Omega) = W^{m,2}(\Omega)$ and $\|\cdot\|_{m,\Omega}$ its norm. For a vector-valued function $\mathbf{u} = (u_1, u_2) \in (H^m(\Omega))^2$, we use the same notation $\|\mathbf{u}\|_{m,\Omega} = \left(\|u_1\|_{m,\Omega}^2 + \|u_2\|_{m,\Omega}^2 \right)^{1/2}$. For a function $\psi \in \mathcal{C}^0(\overline{\Omega})$ we denote by $\|\psi\|_{\mathcal{C}^0(\overline{\Omega})} = \sup_{\mathbf{x}\in\overline{\Omega}} |\psi(\mathbf{x})|$ and if $\psi \in \mathcal{C}^2(\overline{\Omega})$, we use the notation $\|\psi\|_{\mathcal{C}^2(\overline{\Omega})} = \max_{0\le|\alpha|\le 2} \|D^\alpha\psi\|_{\mathcal{C}^0(\overline{\Omega})}$, where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\alpha_1, \alpha_2 \in \mathbb{N}$, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha\psi = \frac{\partial^{|\alpha|}\psi}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}}$. For a vector-valued smooth function $\boldsymbol{\psi} = (\psi_1, \psi_2) \in \left(\mathcal{C}^2(\overline{\Omega})\right)^2$ we use the notation $\|\boldsymbol{\psi}\|_{\mathcal{C}^2(\overline{\Omega})} = \max_{i=1,2} \|\psi_i\|_{\mathcal{C}^2(\overline{\Omega})}$.

According to Boyer, Fabrie [5], Proposition III.2.9, p. 142, since Ω_0^S is bounded with Lipschitz boundary, any $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$ is equal almost everywhere to a Lipschitz continuous function in Ω_0^S, still referred as $\mathbf{u}$, and we have

$$Lip(\mathbf{u}) \le C(\Omega_0^S)\, \|\mathbf{u}\|_{1,\infty,\Omega_0^S}$$

where the constant $C(\Omega_0^S) > 0$ depends on Ω_0^S and

$$Lip(\mathbf{u}) = \sup_{\substack{\mathbf{x}\neq\mathbf{y} \\ \mathbf{x},\mathbf{y}\in\Omega_0^S}} \frac{|\mathbf{u}(\mathbf{x}) - \mathbf{u}(\mathbf{y})|}{|\mathbf{x}-\mathbf{y}|}.$$

The map $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2 \to \det(\mathbf{I} + \nabla\mathbf{u}) \in L^\infty(\Omega_0^S)$ is continuous. Then, for every $0 < \delta < 1$, there exists $0 < \eta_\delta < \frac{1}{C(\Omega_0^S)}$ such that

$$1 - \delta \le \det(\mathbf{I} + \nabla\mathbf{u}) \le 1 + \delta, \quad \text{a.e. } \mathbf{x} \in \Omega_0^S \tag{3.1}$$

for all $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$ that satisfy $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} \le \eta_\delta$. We define

$$B_\delta = \{\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2 ; ||\mathbf{u}||_{1,\infty,\Omega_0^S} \le \eta_\delta,\ \mathbf{u} = 0 \text{ on } \Gamma_D\}. \tag{3.2}$$

We thus get that, for $\mathbf{u} \in B_\delta$, $Lip(\mathbf{u}) < 1$, which gives the injectivity of the map $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$.

Next, we define the characteristic functions $\chi_{\mathbf{u}}^S : D \to \mathbb{R}$ by

$$\chi_{\mathbf{u}}^S(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \Omega_{\mathbf{u}}^S \\ 0, & \mathbf{x} \in D \setminus \Omega_{\mathbf{u}}^S. \end{cases}$$

Now, we present the constitutive relations of the structure and of the fluid. We assume that the structure verifies the linear elasticity equation. The stress tensor of the structure written in the Lagrangian framework is $\lambda^S(\nabla_{\mathbf{X}}\cdot\mathbf{u}) + 2\mu^S\epsilon_{\mathbf{X}}(\mathbf{u})$, where $\lambda^S, \mu^S > 0$ are the Lamé coefficients and $\epsilon_{\mathbf{X}}(\mathbf{u}) = \frac{1}{2}\left(\nabla_{\mathbf{X}}\mathbf{u} + (\nabla_{\mathbf{X}}\mathbf{u})^T\right)$.

Let us introduce the Hilbert spaces

$$\begin{aligned} W^S &= \left\{\mathbf{w}^S \in \left(H^1\left(\Omega_0^S\right)\right)^2;\ \mathbf{w}^S = 0 \text{ on } \Gamma_D\right\},\\ W &= \left(H_0^1(D)\right)^2,\\ Q &= L_0^2(D) = \left\{q \in L^2(D);\ \int_D q\,dx = 0\right\}. \end{aligned}$$

Let us introduce the bi-linear form $a_S : W^S \times W^S \to \mathbb{R}$,

$$a_S\left(\mathbf{u}, \mathbf{w}^S\right) = \int_{\Omega_0^S} \left(\lambda^S(\nabla_{\mathbf{X}}\cdot\mathbf{u})\left(\nabla_{\mathbf{X}}\cdot\mathbf{w}^S\right) + 2\mu^S\epsilon_{\mathbf{X}}(\mathbf{u}) : \epsilon_{\mathbf{X}}\left(\mathbf{w}^S\right)\right) d\mathbf{X}.$$

We assume that the fluid is Newtonian and the Cauchy stress tensor is given by $\sigma^F(\mathbf{v}, p) = -p\,\mathbf{I} + 2\mu^F\epsilon(\mathbf{v})$ where $\mathbf{I}$ is the unit matrix, $\mu^F > 0$ is the viscosity of the fluid and $\epsilon(\mathbf{v}) = \frac{1}{2}\left(\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right)$.

Introduce the notation

$$\begin{aligned} a_F : \left(H^1(D)\right)^2 \times \left(H^1(D)\right)^2 \to \mathbb{R}, &\qquad a_F(\mathbf{v}, \mathbf{w}) = \int_D 2\mu^F\epsilon(\mathbf{v}) : \epsilon(\mathbf{w})\, d\mathbf{x}\\ b_F : W \times Q \to \mathbb{R}, &\qquad b_F(\mathbf{w}, p) = -\int_D (\nabla\cdot\mathbf{w})\, p\, d\mathbf{x}. \end{aligned}$$

Following for example Girault, Raviart [17], the properties below hold:

$$\exists\alpha_F > 0,\ \forall\mathbf{w} \in W,\quad \alpha_F\|\mathbf{w}\|_{1,D}^2 \le a_F(\mathbf{w}, \mathbf{w}) \tag{3.3}$$

$$\exists M_F > 0,\ \forall\mathbf{v}, \mathbf{w} \in W,\quad |a_F(\mathbf{v}, \mathbf{w})| \le M_F\|\mathbf{v}\|_{1,D}\|\mathbf{w}\|_{1,D} \tag{3.4}$$

$$\exists\beta_F > 0,\quad \inf_{q\in Q, q\neq 0}\ \sup_{w\in W, w\neq 0} \frac{b_F(\mathbf{w}, q)}{\|\mathbf{w}\|_{1,D}\|q\|_{0,D}} \ge \beta_F \tag{3.5}$$

$$\exists N_F > 0,\ \forall\mathbf{w} \in W,\ \forall q \in Q,\quad |b_F(\mathbf{w}, q)| \le N_F\|\mathbf{w}\|_{1,D}\|q\|_{0,D} \tag{3.6}$$

We assume that $\mathbf{f}^F \in \left(L^2(D)\right)^2$, $\mathbf{f}^S \in \left(L^2(\Omega_0^S)\right)^2$ and $\mathbf{g} \in \left(H^{1/2}(\partial D)\right)^2$, such that $\mathbf{g} = 0$ on Σ_2 and $\int_{\Sigma_1}\mathbf{g}\cdot\mathbf{n}^F ds = 0$. For a given $\mathbf{u} \in B_\delta$, we define:

- fluid velocity $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on Σ_1, $\mathbf{v}_\epsilon = 0$ on Σ_2,
- fluid pressure $p_\epsilon \in Q$,
- structure displacement $\mathbf{u}_\epsilon \in W^S$,

as the solution of the following weakly coupled system of PDE's:

$$\begin{aligned} a_F(\mathbf{v}_\varepsilon, \mathbf{w}) + b_F(\mathbf{w}, p_\varepsilon) & \\ + \frac{1}{\varepsilon} \int_D \chi^S_{\mathbf{u}} (\mathbf{v}_\varepsilon \cdot \mathbf{w} + \nabla \mathbf{v}_\varepsilon : \nabla \mathbf{w})\, d\mathbf{x} &= \int_D \mathbf{f}^F \cdot \mathbf{w} d\mathbf{x}, \forall \mathbf{w} \in W \quad (3.7) \\ b_F(\mathbf{v}_\varepsilon, q) &= 0, \forall q \in Q \quad (3.8) \end{aligned}$$

$$\begin{aligned} a_S\left(\mathbf{u}_\varepsilon, \mathbf{w}^S\right) &= \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S\, d\mathbf{X} + \int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\varepsilon, p_\varepsilon) \circ \varphi\right) \mathbf{F}^{-T} : \nabla_{\mathbf{X}} \mathbf{w}^S\, d\mathbf{X} \\ &+ \frac{1}{\varepsilon} \int_{\Omega_0^S} J\left((\mathbf{v}_\varepsilon \circ \varphi) \cdot \mathbf{w}^S + (\nabla \mathbf{v}_\varepsilon \circ \varphi) \mathbf{F}^{-T} : \nabla_{\mathbf{X}} \mathbf{w}^S\right) d\mathbf{X} \\ &- \int_{\Omega_0^S} J\left(\mathbf{f}^F \circ \varphi\right) \cdot \mathbf{w}^S\, d\mathbf{X},\ \forall \mathbf{w}^S \in W^S \quad (3.9) \end{aligned}$$

where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$, $\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{u}(\mathbf{X})$, $J(\mathbf{X}) = \det \mathbf{F}(\mathbf{X})$.

For more details of the derivation of the above weak formulation, we refer to Halanay, Murea, Tiba [22]. The sum of the last three terms in (3.9) is equal to the fluid forces acting on the structure. We will prove later that, for a given $\mathbf{u}$, the system (3.7)-(3.8) has an unique solution $\mathbf{v}_\varepsilon$, p_ε. There exists a unique solution $\mathbf{u}_\varepsilon$ of (3.9), see Halanay, Murea, Tiba [24] and $\mathbf{u}_\varepsilon$ has the physical meaning of structural displacement.

Let $\mathcal{P}_\theta : \left(H^1(\Omega_0^S)\right)^2 \to \left(W^{1,\infty}(\Omega_0^S)\right)^2$ be a regularization operator, which will be constructed later, where $\theta > 0$ is a fixed parameter. Define the nonlinear operator $T_\varepsilon^\theta : B_\delta \to \left(W^{1,\infty}(\Omega_0^S)\right)^2$ by $T_\varepsilon^\theta(\mathbf{u}) = \mathcal{P}_\theta(\mathbf{u}_\epsilon)$. The operator T_ε^θ is the composition of three operators:

$$\begin{aligned} &\mathcal{F}_\varepsilon : B_\delta \to \left(\left(H^1(D)\right)^2, Q\right), \quad \mathcal{F}_\varepsilon(\mathbf{u}) = (\mathbf{v}_\varepsilon, p_\varepsilon), \\ &\mathcal{S}_\varepsilon : \left(B_\delta, \left(H^1(D)\right)^2, Q\right) \to W^S, \quad \mathcal{S}_\varepsilon(\mathbf{u}, \mathbf{v}_\varepsilon, p_\varepsilon) = \mathbf{u}_\varepsilon \end{aligned}$$

and $\mathcal{P}_\theta$, more precisely $T_\varepsilon^\theta(\mathbf{u}) = \mathcal{P}_\theta\left(\mathcal{S}_\varepsilon(\mathbf{u}, \mathcal{F}_\varepsilon(\mathbf{u}))\right)$.

In the following, we will prove that T_ε^θ is well defined and that it has at least one fixed point in B_δ.

4 Extension operators

Let D, $\Omega^S_{\mathbf{u}}$ be as in the Section 2. As in Halanay, Murea, Tiba [24], we assume that

$$\begin{array}{c} \partial\Omega^S_{\mathbf{u}} \text{ has the uniform cone property and} \\ \text{the geometry of the cone is independent of } \mathbf{u} \in B_\delta. \end{array} \quad (4.1)$$

In Chenais [8] it is proved that the Lipschitz boundary condition and domains with the uniform cone property are equivalent in a certain sense.

Lemma 1. *We suppose that Σ_2 is an open segment of the Ox_1 axis and $D \subset \mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2;\ x_2 > 0\}$. We assume that Γ_D and $\Gamma_{\mathbf{u}}$, respectively Σ_2 and Σ_1, meet transversally. Then there exists an uniform extension operator*

$$E_1 : \{\mathbf{v} \in \left(H^1\left(\Omega^S_{\mathbf{u}}\right)\right)^2;\ \mathbf{v} = 0 \text{ on } \Gamma_D\} \to \left(H^1_0(D)\right)^2,$$

such that

$$\begin{array}{rcl} E_1(\mathbf{v}) & = & \mathbf{v}, \quad in\ \Omega^S_{\mathbf{u}} \\ \|E_1(\mathbf{v})\|_{1,D} & \leq & K_1 \|\mathbf{v}\|_{1,\Omega^S_{\mathbf{u}}} \end{array}$$

where the constant $K_1 > 0$ *is independent of* $\Omega^S_{\mathbf{u}}$*, but it depends on the geometry of the cone from the assumption (4.1).*

Proof. We have that $\Gamma_D \subset \Sigma_2$. We denote by $sym : \mathbb{R}^2 \to \mathbb{R}^2$ the symmetry operator with respect to the axis Ox_1 given by

$$sym(x_1, x_2) = (x_1, -x_2).$$

Let $\mathbf{v} \in \left(H^1\left(\Omega^S_{\mathbf{u}}\right)\right)^2$ such that $\mathbf{v} = 0$ on Γ_D. It is known that, if $\mathbf{v} \in \left(H^1\left(\Omega^S_{\mathbf{u}}\right)\right)^2$ then $\widehat{\mathbf{v}} = (\mathbf{v} \circ sym) \in \left(H^1\left(sym(\Omega^S_{\mathbf{u}})\right)\right)^2$, see for example Brezis [6], Prop. IX.6, p. 156.

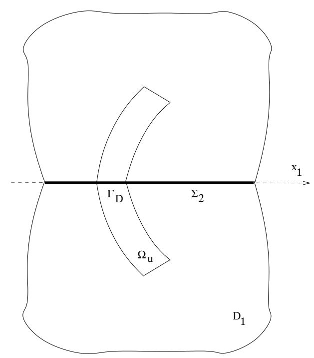

Figure 2: Geometry obtained by symmetry with respect to the axis Ox_1

We define the domain $\Omega_{\mathbf{u}} = \Omega^S_{\mathbf{u}} \cup sym(\Omega^S_{\mathbf{u}}) \cup \Gamma_D$, as in Figure 2 and the function $\widetilde{\mathbf{v}} : \Omega_{\mathbf{u}} \to \mathbb{R}^2$

$$\widetilde{\mathbf{v}} = \begin{cases} \mathbf{v} & \text{in } \Omega^S_{\mathbf{u}} \\ 0 & \text{on } \Gamma_D \\ -\widehat{\mathbf{v}} & \text{in } sym(\Omega^S_{\mathbf{u}}). \end{cases}$$

We have that $\Omega_{\mathbf{u}}$ is bounded open domain. We denote the intersection points $\overline{\Gamma}_D \cap \overline{\Gamma}_{\mathbf{u}} = \{R_1, R_2\}$. Since Γ_D and $\Gamma_{\mathbf{u}}$ meet transversally, then R_1 and R_2 will not be turning points of $\partial\Omega_{\mathbf{u}}$ and consequently Ω has Lipschitz boundary.

But $\mathbf{v} = \widehat{\mathbf{v}} = 0$ on Γ_D, then $\widetilde{\mathbf{v}} \in \left(H^1(\Omega)\right)^2$.

We set $D_1 = D \cup sym(D) \cup \Sigma_2$ which is a bounded open domain. Since Σ_2 and Σ_1 meet transversally, D_1 has Lipschitz boundary. Since $\Omega_{\mathbf{u}} \subset\subset D_1$, from Chenais [8], there exists the operator $e : \left(H^1(\Omega_{\mathbf{u}})\right)^2 \to \left(H^1_0(D_1)\right)^2$ such that

$$\begin{array}{rcl} e(\widetilde{\mathbf{v}}) & = & \widetilde{\mathbf{v}}, \quad \text{in } \Omega_{\mathbf{u}} \\ \|e(\widetilde{\mathbf{v}})\|_{1,D_1} & \leq & K \|\widetilde{\mathbf{v}}\|_{1,\Omega_{\mathbf{u}}} \end{array}$$

where the constant K is independent of $\mathbf{u}$, but it depends on the geometry of the cone from the assumption (4.1).

We can construct the extension operator E_1 as follow:

$$E_1(\mathbf{v}) = \frac{1}{2}\left(e(\widetilde{\mathbf{v}}) - e(\widetilde{\mathbf{v}}) \circ sym\right).$$

We have that $e(\widetilde{\mathbf{v}})$ is in $\left(H_0^1(D_1)\right)^2$ as well as $e(\widetilde{\mathbf{v}}) \circ sym$ and it follows that $E_1(\mathbf{v}) \in \left(H_0^1(D_1)\right)^2$. We observe that $E_1(\mathbf{v}) \circ sym = -E_1(\mathbf{v})$.

Let $\mathcal{C}_0^\infty(D_1)$ denote the space of infinitely differentiable functions with compact support in D_1. Let $\psi_k \in (\mathcal{C}_0^\infty(D_1))^2$ such that $\psi_k \to e(\widetilde{\mathbf{v}})$ in $\left(H_0^1(D_1)\right)^2$. We set $\phi_k = \frac{1}{2}(\psi_k - \psi_k \circ sym)$ and we have that $\phi_k \to E_1(\mathbf{v})$ in $\left(H_0^1(D_1)\right)^2$. For the continuous function $\phi_k : D_1 \to \mathbb{R}^2$ such that $\phi_k(x_1, -x_2) = -\phi_k(x_1, x_2)$ in D_1, we get $\phi_k(x_1, 0) = 0$ for all x_1. Let $\gamma_{\Sigma_2} : \left(H_0^1(D_1)\right)^2 \to \left(H^{1/2}(\Sigma_2)\right)^2$ be the trace on Σ_2. We have $\gamma_{\Sigma_2}(\phi_k) = 0$, for all k. Since γ_{Σ_2} is a linear continuous function, we get $\gamma_{\Sigma_2}(E_1(\mathbf{v})) = 0$, then $E_1(\mathbf{v})_{|D} \in \left(H_0^1(D)\right)^2$.

From construction, we have $\frac{\partial \widehat{v}_i}{\partial x_1} = \frac{\partial v_i}{\partial x_1} \circ sym$ and $\frac{\partial \widehat{v}_i}{\partial x_2} = -\frac{\partial v_i}{\partial x_2} \circ sym$ in $sym(\Omega_{\mathbf{u}}^S)$. It follows

$$\begin{aligned}
\|\widetilde{\mathbf{v}}\|_{1,\Omega_{\mathbf{u}}}^2 &= \int_{\Omega_{\mathbf{u}}} \widetilde{\mathbf{v}} \cdot \widetilde{\mathbf{v}} + \nabla\widetilde{\mathbf{v}} : \nabla\widetilde{\mathbf{v}}\, d\mathbf{x} \\
&= \int_{\Omega_{\mathbf{u}}^S} \mathbf{v} \cdot \mathbf{v} + \nabla\mathbf{v} : \nabla\mathbf{v} d\mathbf{x} + \int_{sym(\Omega_{\mathbf{u}}^S)} \widehat{\mathbf{v}} \cdot \widehat{\mathbf{v}} + \nabla\widehat{\mathbf{v}} : \nabla\widehat{\mathbf{v}}\, d\mathbf{x} \\
&= 2\int_{\Omega_{\mathbf{u}}^S} \mathbf{v} \cdot \mathbf{v} + \nabla\mathbf{v} : \nabla\mathbf{v} d\mathbf{x} = 2\|\mathbf{v}\|_{1,\Omega_{\mathbf{u}}^S}^2
\end{aligned}$$

We have that $\|e(\widetilde{\mathbf{v}})\|_{1,D_1} = \|e(\widetilde{\mathbf{v}}) \circ sym\|_{1,D_1}$, then

$$\begin{aligned}
\|E_1(\mathbf{v})\|_{1,D} &\leq \|E_1(\mathbf{v})\|_{1,D_1} \leq \frac{1}{2}\left(\|e(\widetilde{\mathbf{v}})\|_{1,D_1} + \|e(\widetilde{\mathbf{v}}) \circ sym\|_{1,D_1}\right) \\
&= \|e(\widetilde{\mathbf{v}})\|_{1,D_1} \leq K\|\widetilde{\mathbf{v}}\|_{1,\Omega_{\mathbf{u}}} = \sqrt{2}K\|\mathbf{v}\|_{1,\Omega_{\mathbf{u}}^S}
\end{aligned}$$

From $e(\widetilde{\mathbf{v}}) = \widetilde{\mathbf{v}}$ in $\Omega_{\mathbf{u}}$, we obtain $e(\widetilde{\mathbf{v}}) = \mathbf{v}$ in $\Omega_{\mathbf{u}}^S$ and $e(\widetilde{\mathbf{v}}) = -\widehat{\mathbf{v}}$ in $sym(\Omega_{\mathbf{u}}^S)$ which gives $e(\widetilde{\mathbf{v}}) \circ sym = -\widehat{\mathbf{v}} \circ sym = -\mathbf{v}$ in $\Omega_{\mathbf{u}}^S$. Finnaly, we get $E_1(\mathbf{v}) = \mathbf{v}$ in $\Omega_{\mathbf{u}}^S$. □

Theorem 1. *Under the hypotheses of Lemma 1, there exists an uniform extension operator*

$$E : \{\mathbf{v} \in \left(H^1\left(\Omega_{\mathbf{u}}^S\right)\right)^2;\ \nabla\cdot\mathbf{v} = 0 \text{ in } \Omega_{\mathbf{u}}^S,\ \mathbf{v} = 0 \text{ on } \Gamma_D\} \to \left(H_0^1(D)\right)^2,$$

such that

$$\begin{aligned}
\nabla \cdot E(\mathbf{v}) &= 0, \quad \text{in } D \\
E(\mathbf{v}) &= \mathbf{v}, \quad \text{in } \Omega_{\mathbf{u}}^S \\
\|E(\mathbf{v})\|_{1,D} &\leq K_2\|\mathbf{v}\|_{1,\Omega_{\mathbf{u}}^S}
\end{aligned}$$

where the constant $K_2 > 0$ is independent of $\Omega_{\mathbf{u}}^S$, but it depends on the geometry of the cone from the assumption (4.1).

Proof. We follow the Lemma 5.1 from Halanay, Murea, Tiba [24] or Corollary 3.1, Chapter III, p. 136 from Galdi [15]. Let Z be an open rectangle such that $\Omega_{\mathbf{u}}^S \subset Z \subset D$, the bottom side of the rectangle is included in Σ_2 and $dist(Z, \Sigma_1) > 0$, see Figure 3.

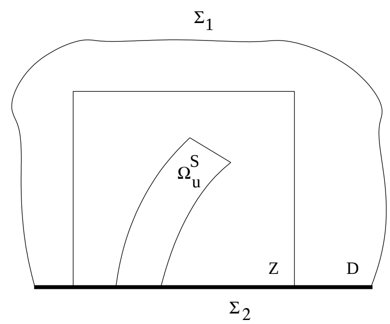

Figure 3: Configuration of Z.

We can apply Lemma 1, then there exists

$$E_1 : \{\mathbf{v} \in \left(H^1\left(\Omega_{\mathbf{u}}^S\right)\right)^2;\ \mathbf{v} = 0 \text{ on } \Gamma_D\} \rightarrow \left(H_0^1(Z)\right)^2$$

such that

$$\begin{aligned} E_1(\mathbf{v}) &= \mathbf{v}, \quad \text{in } \Omega_{\mathbf{u}}^S \\ \|E_1(\mathbf{v})\|_{1,Z} &\leq K_1 \|\mathbf{v}\|_{1,\Omega_{\mathbf{u}}^S}. \end{aligned}$$

We denote

$$\widetilde{E_1(\mathbf{v})} = \begin{cases} E_1(\mathbf{v}), & \text{in } Z \\ 0, & \text{in } D \setminus \overline{Z}. \end{cases}$$

Since $E_1(\mathbf{v})$ belongs to $\left(H_0^1(Z)\right)^2$, then $\widetilde{E_1(\mathbf{v})}$ is in $\left(H_0^1(D)\right)^2$.

Now, we solve the Bogowskii problem in $D \setminus \overline{\Omega}_{\mathbf{u}}^S$, see for example Galdi [15] Theorem 3.1, p. 129. There exists $\mathbf{w} \in \left(H_0^1\left(D \setminus \overline{\Omega}_{\mathbf{u}}^S\right)\right)^2$ such that

$$\begin{aligned} \nabla \cdot \mathbf{w} &= \nabla \cdot \widetilde{E_1(\mathbf{v})}, \quad \text{in } D \setminus \overline{\Omega}_{\mathbf{u}}^S \\ \mathbf{w} &= 0, \quad \text{on the boundary of } D \setminus \overline{\Omega}_{\mathbf{u}}^S \\ \|\mathbf{w}\|_{1,D\setminus\overline{\Omega}_{\mathbf{u}}^S} &\leq K \left\|\nabla \cdot \widetilde{E_1(\mathbf{v})}\right\|_{0,D\setminus\overline{\Omega}_{\mathbf{u}}^S} \end{aligned}$$

where the constant $K > 0$ is independent of $\Omega_{\mathbf{u}}^S$. In fact, if a bounded domain is the union of a star-shaped domains with respect to every point of some balls, then the constant K depends only on the radius of the balls, see Theorem 3.1, p. 129, Galdi [15]. In the same reference, Lemma 3.2, p. 39, it in proved that a Lipschitz domain is the union of this kind of star-shaped domains. We could prove that the radius of the balls does not change under small perturbation of a domain.

We introduce

$$\widetilde{\mathbf{w}} = \begin{cases} \mathbf{w}, & \text{in } D \setminus \overline{\Omega}_{\mathbf{u}}^S \\ 0, & \text{in } \Omega_{\mathbf{u}}^S \end{cases}$$

and we have that $\widetilde{\mathbf{w}} \in \left(H_0^1(D)\right)^2$.

The uniform extension operator with free divergence is defined by $E(\mathbf{v}) = \widetilde{E_1(\mathbf{v})} - \widetilde{\mathbf{w}}$ which belongs to $\left(H_0^1(D)\right)^2$. Moreover, it verifies

$$E(\mathbf{v}) = \begin{cases} \widetilde{E_1(\mathbf{v})} = E_1(\mathbf{v}) = \mathbf{v}, & \text{in } \Omega_{\mathbf{u}}^S \\ \widetilde{E_1(\mathbf{v})} - \mathbf{w}, & \text{in } D \setminus \overline{\Omega}_{\mathbf{u}}^S. \end{cases}$$

The rest is as in the Lemma 5.1 from Halanay, Murea, Tiba [24]. ◻

Remark 1. *In the case when the boundary Σ_2 is not a straight segment, we can follow the procedure presented in Evans [13], Appendix C, p. 711. We can suppose that Σ_2 is the graph of a real function γ of class $\mathcal{C}^1$. We introduce the applications $\Phi(x_1, x_2) = (x_1, x_2 - \gamma(x_1))$ and $\Psi(y_1, y_2) = (y_1, y_2 + \gamma(y_1))$. We have that $\Phi^{-1} = \Psi$ and $\det(\nabla\Phi(\mathbf{x})) = 1$ as well as $\det(\nabla\Psi(\mathbf{y})) = 1$. We have that $\Phi(\Sigma_2)$ is a straight segment and we can apply the Lemma 1 for $\Phi(\Omega_{\mathbf{u}}^S)$ and $\Phi(D)$. The conclusion of Lemma 1 remains true.*

Remark 2. *Also, Theorem 1 holds if Σ_2 is not a straight segment. To prove that, we can use Remark 1 to get the extension operator E_1 without divergence free. The second part of the proof is the same as in Theorem 1 based on the solution of the Bogowskii problem which does not use the straightness of Σ_2.*

5 Estimations

Proposition 1. *We preserve the setting of Lemma 1. Assume that $\mathbf{f}^F \in \left(L^2(D)\right)^2$, $\mathbf{g} \in \left(H^{1/2}(\partial D)\right)^2$, such that $\mathbf{g} = 0$ on Σ_2, $\int_{\Sigma_1} \mathbf{g} \cdot \mathbf{n}^F ds = 0$ and $\mathbf{u} \in B_\delta$. There exists a unique solution of (3.7)-(3.8) such that $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on Σ_1, $\mathbf{v}_\epsilon = 0$ on Σ_2 and $p_\varepsilon \in Q$. Moreover, there exists a constant C_1 independent of $\varepsilon > 0$ and $\mathbf{u} \in B_\delta$, such that*

$$\|\mathbf{v}_\varepsilon\|_{1,D} + \|p_\varepsilon\|_{0,D} \leq C_1 \left(\left\|\mathbf{f}^F\right\|_{0,D} + \|\mathbf{g}\|_{1/2,\Sigma_1} \right). \tag{5.1}$$

Proof. Let Z be an open rectangle such that $\Omega_{\mathbf{u}}^S \subset Z \subset D$, the bottom side of the rectangle is included in Σ_2 and $dist(Z, \Sigma_1) > 0$, see Figure 3. Following Galdi [15], there exists $\mathbf{v}_g \in \left(H^1\left(D \setminus \overline{Z}\right)\right)^2$, such that $\nabla \cdot \mathbf{v}_g = 0$ in $D \setminus \overline{Z}$, $\mathbf{v}_g = \mathbf{g}$ on Σ_1 and $\mathbf{v}_g = 0$ on $\partial\left(D \setminus \overline{Z}\right) \setminus \Sigma_1$. We extend $\mathbf{v}_g$ by zero in Z and we get $\mathbf{v}_g \in \left(H^1(D)\right)^2$ such that $\nabla \cdot \mathbf{v}_g = 0$ in D, $\mathbf{v}_g = \mathbf{g}$ on Σ_1 and $\mathbf{v}_g = 0$ on Σ_2. Moreover, we have

$$\|\mathbf{v}_g\|_{1,D} \leq K_3 \|\mathbf{g}\|_{1/2,\Sigma_1}.$$

From (3.7) and using that $\mathbf{v}_g = 0$ in $\Omega_{\mathbf{u}}^S$, we obtain

$$a_F(\mathbf{v}_\varepsilon - \mathbf{v}_g, \mathbf{w}) + b_F(\mathbf{w}, p_\varepsilon) + \frac{1}{\varepsilon}(\mathbf{v}_\varepsilon - \mathbf{v}_g, \mathbf{w})_{1,\Omega_{\mathbf{u}}^S} = \int_D \mathbf{f}^F \cdot \mathbf{w} d\mathbf{x} - a_F(\mathbf{v}_g, \mathbf{w}) \tag{5.2}$$

for all $\mathbf{w} \in W$. From (3.8), we get

$$b_F(\mathbf{v}_\varepsilon - \mathbf{v}_g, q) = 0, \forall q \in Q. \tag{5.3}$$

From the Babuska-Brezzi theorem, see for example Girault, Raviart [17], the problem (5.2)-(5.3) has a unique solution $\mathbf{v}_\varepsilon - \mathbf{v}_g \in W$ and $p_\varepsilon \in Q$.

Putting $\mathbf{w} = \mathbf{v}_\varepsilon - \mathbf{v}_g$ in (5.2), we get

$$\begin{aligned} a_F(\mathbf{v}_\varepsilon - \mathbf{v}_g, \mathbf{v}_\varepsilon - \mathbf{v}_g) + \frac{1}{\varepsilon}(\mathbf{v}_\varepsilon - \mathbf{v}_g, \mathbf{v}_\varepsilon - \mathbf{v}_g)_{1,\Omega^S_{\mathbf{u}}} &= \int_D \mathbf{f}^F \cdot (\mathbf{v}_\varepsilon - \mathbf{v}_g) d\mathbf{x} \\ &\quad - a_F(\mathbf{v}_g, \mathbf{v}_\varepsilon - \mathbf{v}_g). \end{aligned}$$

Using the ellipticity, the continuity of a_F and the fact that $\mathbf{v}_g = 0$ in $\Omega^S_{\mathbf{u}}$, it follows

$$\begin{aligned} \alpha_F \|\mathbf{v}_\varepsilon - \mathbf{v}_g\|^2_{1,D} + \frac{1}{\varepsilon}\|\mathbf{v}_\varepsilon\|^2_{1,\Omega^S_{\mathbf{u}}} &\le \int_D \mathbf{f}^F \cdot (\mathbf{v}_\varepsilon - \mathbf{v}_g) d\mathbf{x} - a_F(\mathbf{v}_g, \mathbf{v}_\varepsilon - \mathbf{v}_g) \\ &\le \left\|\mathbf{f}^F\right\|_{0,D} \|\mathbf{v}_\varepsilon - \mathbf{v}_g\|_{0,D} + M_F \|\mathbf{v}_g\|_{1,D} \|\mathbf{v}_\varepsilon - \mathbf{v}_g\|_{1,D} \\ &\le \left(\left\|\mathbf{f}^F\right\|_{0,D} + M_F \|\mathbf{v}_g\|_{1,D}\right) \|\mathbf{v}_\varepsilon - \mathbf{v}_g\|_{1,D} \end{aligned}$$

and then

$$\|\mathbf{v}_\varepsilon - \mathbf{v}_g\|_{1,D} \le \frac{1}{\alpha_F}\left(\left\|\mathbf{f}^F\right\|_{0,D} + M_F K_3 \|\mathbf{g}\|_{1/2,\Sigma_1}\right).$$

From the triangle inequality, we obtain

$$\|\mathbf{v}_\varepsilon\|_{1,D} \le \|\mathbf{v}_\varepsilon - \mathbf{v}_g\|_{1,D} + \|\mathbf{v}_g\|_{1,D} \le K_4\left(\left\|\mathbf{f}^F\right\|_{0,D} + \|\mathbf{g}\|_{1/2,\Sigma_1}\right). \tag{5.4}$$

The next steps are as in Halanay, Murea, Tiba [24]. Putting $\mathbf{w} = E(\mathbf{v}_\varepsilon)$ in (3.7), where E is the extension operator defined in Lemma 1, we get

$$a_F(\mathbf{v}_\varepsilon, E(\mathbf{v}_\varepsilon)) + \frac{1}{\varepsilon}\int_D \chi^S_{\mathbf{u}}\left(\mathbf{v}_\varepsilon \cdot E(\mathbf{v}_\varepsilon) + \nabla \mathbf{v}_\varepsilon : \nabla E(\mathbf{v}_\varepsilon)\right) d\mathbf{x} = \int_D \mathbf{f}^F \cdot E(\mathbf{v}_\varepsilon) d\mathbf{x}$$

then

$$\frac{1}{\varepsilon}\int_{\Omega^S_{\mathbf{u}}}\left(\mathbf{v}_\varepsilon \cdot E(\mathbf{v}_\varepsilon) + \nabla \mathbf{v}_\varepsilon : \nabla E(\mathbf{v}_\varepsilon)\right) d\mathbf{x} = \int_D \mathbf{f}^F \cdot E(\mathbf{v}_\varepsilon) d\mathbf{x} - a_F(\mathbf{v}_\varepsilon, E(\mathbf{v}_\varepsilon))$$

and consequently

$$\frac{1}{\varepsilon}\|\mathbf{v}_\varepsilon\|^2_{1,\Omega^S_{\mathbf{u}}} \le \left\|\mathbf{f}^F\right\|_{0,D} \|E(\mathbf{v}_\varepsilon)\|_{0,D} + M_F \|\mathbf{v}_\varepsilon\|_{1,D} \|E(\mathbf{v}_\varepsilon)\|_{1,D}.$$

It follows using Lemma 1 and (5.4)

$$\frac{1}{\varepsilon}\|\mathbf{v}_\varepsilon\|_{1,\Omega^S_{\mathbf{u}}} \le \left(\left\|\mathbf{f}^F\right\|_{0,D} + M_F \|\mathbf{v}_\varepsilon\|_{1,D}\right) K_2 \le K_5\left(\left\|\mathbf{f}^F\right\|_{0,D} + \|\mathbf{g}\|_{1/2,\Sigma_1}\right). \tag{5.5}$$

Now, we can estimate the fluid pressure. From (3.7), we have

$$\begin{aligned} |b_F(\mathbf{w},p_\varepsilon)| &\le \left|\int_D \mathbf{f}^F\cdot\mathbf{w}d\mathbf{x}\right| + |a_F(\mathbf{v}_\varepsilon,\mathbf{w})| + \left|\frac{1}{\varepsilon}\int_D \chi^S_{\mathbf{u}}\left(\mathbf{v}_\varepsilon\cdot\mathbf{w}+\nabla\mathbf{v}_\varepsilon:\nabla\mathbf{w}\right)d\mathbf{x}\right| \\ &\le \left\|\mathbf{f}^F\right\|_{0,D}\|\mathbf{w}\|_{0,D} + M_F\|\mathbf{v}_\varepsilon\|_{1,D}\|\mathbf{w}\|_{1,D} + \frac{1}{\varepsilon}\|\mathbf{v}_\varepsilon\|_{1,\Omega^S_{\mathbf{u}}}\|\mathbf{w}\|_{1,\Omega^S_{\mathbf{u}}} \\ &\le \left(\left\|\mathbf{f}^F\right\|_{0,D} + M_F\|\mathbf{v}_\varepsilon\|_{1,D} + \frac{1}{\varepsilon}\|\mathbf{v}_\varepsilon\|_{1,\Omega^S_{\mathbf{u}}}\right)\|\mathbf{w}\|_{1,D}. \end{aligned}$$

Taking into account the estimations (5.4) and (5.5), we get

$$\forall\mathbf{w}\in W,\quad |b_F(\mathbf{w},p_\varepsilon)| \le K_6\left(\left\|\mathbf{f}^F\right\|_{0,D} + \|\mathbf{g}\|_{1/2,\Sigma_1}\right)\|\mathbf{w}\|_{1,D}.$$

The *inf-sup* condition of b_F implies that

$$\beta_F\|p_\varepsilon\|_{0,D} \le \sup_{w\in W, w\neq 0}\frac{b_F(\mathbf{w},p_\varepsilon)}{\|\mathbf{w}\|_{1,D}} \le K_6\left(\left\|\mathbf{f}^F\right\|_{0,D} + \|\mathbf{g}\|_{1/2,\Sigma_1}.\right) \tag{5.6}$$

From (5.4) and (5.6), we get (5.1). ∎

Proposition 2. *We assume that* $\mathbf{f}^S \in \left(L^2(\Omega^S_0)\right)^2$, $\mathbf{f}^F \in \left(L^2(D)\right)^2$ *and* $\mathbf{u}\in B_\delta$. *If* $\mathbf{v}_\varepsilon$ *and* p_ε *are solutions of (3.7)-(3.8), then the problem (3.9) has a unique solution* $\mathbf{u}_\varepsilon \in W^S$ *and there exists a constant* C_2 *independent of* $\varepsilon > 0$ *and* $\mathbf{u}\in B_\delta$*, such that*

$$\|\mathbf{u}_\varepsilon\|_{1,\Omega^S_0} \le C_2\left(\left\|\mathbf{f}^S\right\|_{0,\Omega^S_0} + \|\mathbf{v}_\varepsilon\|_{1,D} + \|p_\varepsilon\|_{0,D} + \left\|\mathbf{f}^F\right\|_{0,D}\right). \tag{5.7}$$

The proof is as in Halanay, Murea, Tiba [24].

Proposition 3. *There exists a family of linear regularization operators depending on a parameter* $\theta > 0$

$$\mathcal{P}_\theta : \left\{\mathbf{u}\in\left(H^1(\Omega^S_0)\right)^2;\ \mathbf{u}=0 \text{ on } \Gamma_D\right\} \to \left(\mathcal{C}^2(\overline{\Omega}^S_0)\right)^2$$

such that:
i) there exists $C_3(\theta) > 0$,

$$\forall\mathbf{u}\in\left(H^1(\Omega^S_0)\right)^2,\ \mathbf{u}=0 \text{ on } \Gamma_D,\quad \|\mathcal{P}_\theta(\mathbf{u})\|_{\mathcal{C}^2(\overline{\Omega}^S_0)} \le C_3(\theta)\|\mathbf{u}\|_{1,\Omega^S_0}. \tag{5.8}$$

Moreover, $\mathcal{P}_\theta$ *is compact in* $\left(W^{1,\infty}(\Omega^S_0)\right)^2$.
ii) $\mathcal{P}_\theta(\mathbf{u}) = 0$ *on* Γ_D.

Proof. i) Let $\mathbf{u}\in\left(H^1(\Omega^S_0)\right)^2$ such that $\mathbf{u}=0$ on Γ_D. We have that $\widehat{\mathbf{u}} = (\mathbf{u}\circ sym) \in \left(H^1\left(sym(\Omega^S_0)\right)\right)^2$ and we define the function $\widetilde{\mathbf{u}}:\mathbb{R}^2\to\mathbb{R}^2$

$$\widetilde{\mathbf{u}} = \begin{cases} \mathbf{u} & \text{in } \Omega^S_0 \\ \widehat{\mathbf{u}} & \text{in } sym(\Omega^S_0) \\ 0 & \text{otherwise.} \end{cases}$$

We have that $\widetilde{\mathbf{u}}$ is in $\left(L^2(\mathbb{R}^2)\right)^2$. Let $\theta > 0$ be a fixed parameter. We define the regularization function

$$\widetilde{\mathbf{u}}_\theta(\mathbf{x}) = \theta^{-2} \int_{\mathbb{R}^2} j\left(\frac{\mathbf{x}-\mathbf{y}}{\theta}\right) \widetilde{\mathbf{u}}(\mathbf{y}) d\mathbf{y}$$

using the standard mollifier j, see Adams [1], p. 29. We set $j_\theta(\mathbf{x}) = \theta^{-2} j\left(\frac{\mathbf{x}}{\theta}\right)$. Following the same reference, the Lemma 2.18, p. 29, we obtain that $\widetilde{\mathbf{u}}_\theta \in \mathcal{C}^\infty(\mathbb{R}^2)$ and

$$D^\alpha \widetilde{\mathbf{u}}_\theta(\mathbf{x}) = \int_{\mathbb{R}^2} D^\alpha j_\theta\left(\mathbf{x}-\mathbf{y}\right) \widetilde{\mathbf{u}}(\mathbf{y}) d\mathbf{y}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index, $\alpha_1, \alpha_2 \in \mathbb{N}$, $|\alpha| = \alpha_1 + \alpha_2$ and $D^\alpha j = \frac{\partial^{|\alpha|} j}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2}}$. Using Cauchy-Schwarz inequality for each component of $\widetilde{\mathbf{u}} = (\widetilde{u}_1, \widetilde{u}_2)$, we get

$$\forall \mathbf{x} \in \mathbb{R}^2, \quad \left| \int_{\mathbb{R}^2} D^\alpha j_\theta\left(\mathbf{x}-\mathbf{y}\right) \widetilde{u}_i(\mathbf{y}) d\mathbf{y} \right| \quad \leq \quad \left\| D^\alpha j_\theta \right\|_{0,\mathbb{R}^2} \left\| \widetilde{u}_i \right\|_{0,\mathbb{R}^2}$$

then

$$\sup_{\mathbf{x} \in \mathbb{R}^2} \left| \int_{\mathbb{R}^2} D^\alpha j_\theta\left(\mathbf{x}-\mathbf{y}\right) \widetilde{u}_i(\mathbf{y}) d\mathbf{y} \right| \quad \leq \quad \left\| D^\alpha j_\theta \right\|_{0,\mathbb{R}^2} \left\| \widetilde{u}_i \right\|_{0,\mathbb{R}^2}$$

which implies

$$\begin{aligned} \left\| D^\alpha \widetilde{\mathbf{u}}_\theta \right\|_{\mathcal{C}^0(\mathbb{R}^2)} &\leq \left\| D^\alpha j_\theta \right\|_{0,\mathbb{R}^2} \max_{i=1,2} \left\| \widetilde{u}_i \right\|_{0,\mathbb{R}^2} \\ &\leq \left\| D^\alpha j_\theta \right\|_{0,\mathbb{R}^2} \left\| \widetilde{\mathbf{u}} \right\|_{0,\mathbb{R}^2} = \left\| D^\alpha j_\theta \right\|_{0,\mathbb{R}^2} \sqrt{2} \left\| \mathbf{u} \right\|_{0,\Omega_0^S}. \end{aligned}$$

We have $D^\alpha j_\theta(\mathbf{x}) = \theta^{-2-|\alpha|}(D^\alpha j)\left(\frac{\mathbf{x}}{\theta}\right)$, then $\left\| D^\alpha j_\theta \right\|_{0,\mathbb{R}^2} = \theta^{-1-|\alpha|} \left\| D^\alpha j \right\|_{0,\mathbb{R}^2}$. We deduce that

$$\begin{aligned} \left\| \widetilde{\mathbf{u}}_\theta \right\|_{\mathcal{C}^2(\mathbb{R}^2)} &= \max_{0 \leq |\alpha| \leq 2} \left\| D^\alpha \widetilde{\mathbf{u}}_\theta \right\|_{\mathcal{C}^0(\mathbb{R}^2)} \\ &\leq \max_{0 \leq |\alpha| \leq 2} \left(\theta^{-1-|\alpha|} \left\| D^\alpha j \right\|_{0,\mathbb{R}^2} \right) \sqrt{2} \left\| \mathbf{u} \right\|_{0,\Omega_0^S}. \end{aligned}$$

Using the precedent inequality, we get

$$\left\| \widetilde{\mathbf{u}}_\theta \right\|_{\mathcal{C}^2(\mathbb{R}^2)} \leq C_3(\theta) \left\| \mathbf{u} \right\|_{0,\Omega_0^S}$$

where $C_3(\theta) = \max_{0 \leq |\alpha| \leq 2} \left(\theta^{-1-|\alpha|} \left\| D^\alpha j \right\|_{0,\mathbb{R}^2} \right) \sqrt{2}$. We define $\mathcal{P}_\theta(\mathbf{u})$ the restriction of $\widetilde{\mathbf{u}}_\theta$ to $\overline{\Omega}_0^S$ and we have that $\mathcal{P}_\theta(\mathbf{u}) \in \left(\mathcal{C}^2(\overline{\Omega}_0^S)\right)^2$. Moreover, we deduce

$$\left\| \mathcal{P}_\theta(\mathbf{u}) \right\|_{\mathcal{C}^2(\overline{\Omega}_0^S)} \leq \left\| \widetilde{\mathbf{u}}_\theta \right\|_{\mathcal{C}^2(\mathbb{R}^2)} \leq C_3(\theta) \left\| \mathbf{u} \right\|_{0,\Omega_0^S}$$

which gives (5.8).

The embedding $\mathcal{C}^2(\overline{\Omega}_0^S) \subset \mathcal{C}^1(\overline{\Omega}_0^S)$ is compact (see Queffélec, Zuily [30], Prop. II.5, p. 275). Consequently, $\mathcal{P}_\theta$ is compact in $\left(\mathcal{C}^1(\overline{\Omega}_0^S)\right)^2$. The injection $\left(\mathcal{C}^1(\overline{\Omega}_0^S)\right)^2 \subset \left(W^{1,\infty}(\Omega_0^S)\right)^2$ is linear and continuous, consequently, the operator $\mathcal{P}_\theta$ is compact in $\left(W^{1,\infty}(\Omega_0^S)\right)^2$, too.

ii) It remains to show that $\mathcal{P}_\theta(\mathbf{u}) = 0$ on Γ_D. Using the change of variable formula $(y_1, y_2) \to (y_1, -y_2)$, we obtain

$$\begin{aligned}\widetilde{\mathbf{u}}_\theta(x_1, -x_2) &= \theta^{-2} \int_{\mathbb{R}^2} j\left(\frac{x_1 - y_1}{\theta}, \frac{-x_2 - y_2}{\theta}\right) \widetilde{\mathbf{u}}(y_1, y_2) dy_1 dy_2 \\ &= -\theta^{-2} \int_{\mathbb{R}^2} j\left(\frac{x_1 - y_1}{\theta}, \frac{-x_2 + y_2}{\theta}\right) \widetilde{\mathbf{u}}(y_1, -y_2) dy_1 dy_2\end{aligned}$$

and employing the identities $\widetilde{\mathbf{u}}(y_1, -y_2) = \widetilde{\mathbf{u}}(y_1, y_2)$ and $j(z_1, -z_2) = j(z_1, z_2)$ with $z_1 = \frac{x_1 - y_1}{\theta}$, $z_2 = \frac{x_2 - y_2}{\theta}$, we get

$$\begin{aligned}& -\theta^{-2} \int_{\mathbb{R}^2} j\left(\frac{x_1 - y_1}{\theta}, \frac{-x_2 + y_2}{\theta}\right) \widetilde{\mathbf{u}}(y_1, -y_2) dy_1 dy_2 \\ =& -\theta^{-2} \int_{\mathbb{R}^2} j\left(\frac{x_1 - y_1}{\theta}, \frac{x_2 - y_2}{\theta}\right) \widetilde{\mathbf{u}}(y_1, y_2) dy_1 dy_2 \\ =& -\widetilde{\mathbf{u}}_\theta(x_1, x_2)\end{aligned}$$

therefore $\widetilde{\mathbf{u}}_\theta(x_1, -x_2) = -\widetilde{\mathbf{u}}_\theta(x_1, x_2)$, then $\widetilde{\mathbf{u}}_\theta(x_1, 0) = 0$ and $\mathcal{P}_\theta(\mathbf{u}) = 0$ on Γ_D. □

Remark 3. *The regularization operator depends on the parameter θ and the mollifier j, but in order to simplify the notations, we use $\mathcal{P}_\theta(\mathbf{u})$ in place of $\mathcal{P}_{\theta,j}(\mathbf{u})$. The value of $C_3(\theta) > 0$ is not bounded when $\theta > 0$ goes to zero. We assume that θ is fixed in the following.*

Corollary 1. *If $\mathbf{f}^F$, $\mathbf{g}$ and $\mathbf{f}^S$ verify*

$$C_3(\theta) C_2 \left((C_1 + 1) \left\| \mathbf{f}^F \right\|_{0,D} + C_1 \left\| \mathbf{g} \right\|_{1/2,\Sigma_1} + \left\| \mathbf{f}^S \right\|_{0,\Omega_0^S} \right) \le \eta_\delta,$$

where C_1, C_2, $C_3(\theta)$ are the constants from the Propositions 1, 2, 3, then the operator T_ε^θ has at least one fixed point in B_δ.

Proof. Using the Propositions 1, 2, 3, it follows easily that $T_\varepsilon^\theta(B_\delta) \subset B_\delta$. The nonlinear operator T_ε^θ is the composition of three operators: the fluid operator $\mathcal{F}_\varepsilon(\mathbf{u}) = (\mathbf{v}_\varepsilon, p_\varepsilon)$ defined by the equations (3.7)-(3.8), the structure operator $\mathcal{S}_\varepsilon(\mathbf{u}, \mathbf{v}_\varepsilon, p_\varepsilon) = \mathbf{u}_\varepsilon$ defined by the equation (3.9) and the regularization operator $\mathcal{P}_\theta$, more precisely $T_\varepsilon^\theta(\mathbf{u}) = \mathcal{P}_\theta\left(\mathcal{S}_\varepsilon\left(\mathbf{u}, \mathcal{F}_\varepsilon(\mathbf{u})\right)\right)$. As in Halanay, Murea, Tiba [24], we can prove that $\mathcal{F}_\varepsilon$ and $\mathcal{S}_\varepsilon$ are continuous, then T_ε^θ is continuous, also. Since $\mathcal{P}_\theta$ is compact, then T_ε^θ is compact and by Schauder fixed point theorem, there exists at least a fixed point in B_δ. □

Let $\widehat{\mathbf{u}}_\varepsilon^\theta$ be a fixed point of T_ε^θ. We use the notations

$$\mathcal{F}_\varepsilon(\widehat{\mathbf{u}}_\varepsilon^\theta) = (\mathbf{v}_\varepsilon^\theta, p_\varepsilon^\theta), \quad \mathcal{S}_\varepsilon\left(\widehat{\mathbf{u}}_\varepsilon^\theta, \mathbf{v}_\varepsilon^\theta, p_\varepsilon^\theta\right) = \mathbf{u}_\varepsilon^\theta.$$

We have that

$$\widehat{\mathbf{u}}_\varepsilon^\theta = T_\varepsilon^\theta(\widehat{\mathbf{u}}_\varepsilon^\theta) = \mathcal{P}_\theta\left(\mathcal{S}_\varepsilon\left(\widehat{\mathbf{u}}_\varepsilon^\theta, \mathcal{F}_\varepsilon(\widehat{\mathbf{u}}_\varepsilon^\theta)\right)\right) = \mathcal{P}_\theta\left(\mathcal{S}_\varepsilon\left(\widehat{\mathbf{u}}_\varepsilon^\theta, \mathbf{v}_\varepsilon^\theta, p_\varepsilon^\theta\right)\right) = \mathcal{P}_\theta\left(\mathbf{u}_\varepsilon^\theta\right)$$

then $\widehat{\mathbf{u}}_\varepsilon^\theta = \mathcal{P}_\theta(\mathbf{u}_\varepsilon^\theta)$. The fixed point $\widehat{\mathbf{u}}_\varepsilon^\theta \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$ has the physical meaning of structural displacement like $\mathbf{u}_\varepsilon^\theta \in \left(H^1(\Omega_0^S)\right)^2$, but the smoothness of $\widehat{\mathbf{u}}_\varepsilon^\theta$ allow us to define the deformed structure domain. From Evans [13], Theorem 1, p. 264, we have $\mathcal{P}_\theta(\mathbf{u}_\varepsilon^\theta) \to \mathbf{u}_\varepsilon^\theta$ strongly in $\left(H_{loc}^1(\Omega_0^S)\right)^2$ as $\theta \to 0$, consequently, for small θ, $\widehat{\mathbf{u}}_\varepsilon^\theta$ is close to $\mathbf{u}_\varepsilon^\theta$.

We can solve numerically the problem by using the fixed point iterations as in Halanay, Murea, Tiba [22] and Murea, Halanay [25], or by a quasi-Newton iterative method as in Halanay, Murea [23].

We notice that $\left\|\mathbf{v}_\varepsilon^\theta\right\|_{1,D}$, $\left\|p_\varepsilon^\theta\right\|_{0,D}$, $\left\|\mathbf{u}_\varepsilon^\theta\right\|_{1,\Omega_0^S}$, corresponding to the fixed point $\widehat{\mathbf{u}}_\varepsilon^\theta$ of T_ε^θ, are bounded independent of ε. Then, there exists $\mathbf{v}_*^\theta \in \left(H^1(D)\right)^2$, $p_*^\theta \in L^2(D)$, $\mathbf{u}_*^\theta \in \left(H^1(\Omega_0^S)\right)^2$ and, on a sub-sequence, we have $\mathbf{v}_\varepsilon^\theta \to \mathbf{v}_*^\theta$ weakly in $\left(H^1(D)\right)^2$, $p_\varepsilon^\theta \to p_*^\theta$ weakly in $L^2(D)$, $\mathbf{u}_\varepsilon^\theta \to \mathbf{u}_*^\theta$ weakly in $\left(H^1(\Omega_0^S)\right)^2$. Since $\mathcal{P}_\theta$ is compact, then $\mathcal{P}_\theta(\mathbf{u}_\varepsilon^\theta) \to \mathcal{P}_\theta(\mathbf{u}_*^\theta)$ strongly in $\left(W^{1,\infty}(\Omega_0^S)\right)^2$. To simplify the notation, we set

$$\widehat{\mathbf{u}}_*^\theta = \mathcal{P}_\theta(\mathbf{u}_*^\theta).$$

As in Dautray, Lions [11], chap. VII, page 1241, there exists $\mathbf{j}_S \in \left(H_{00}^{1/2}(\Gamma_0)\right)'$ defined by

$$\left\langle \mathbf{j}_S, \gamma_{\Gamma_0}(\mathbf{w}^S)\right\rangle_{\Gamma_0} = a_S\left(\mathbf{u}_*^\theta, \mathbf{w}^S\right) - \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S \, d\mathbf{X}, \tag{5.9}$$

for all $\mathbf{w}^S \in \left(H^1(\Omega_0^S)\right)^2$, such that $\mathbf{w}^S = 0$ on Γ_D, where $\langle\cdot,\cdot\rangle_{\Gamma_0}$ is the duality $\left(H_{00}^{1/2}(\Gamma_0)\right)'$, $H_{00}^{1/2}(\Gamma_0)$ and γ_{Γ_0} is the trace on Γ_0. We can interpret $\mathbf{j}_S$ as $\sigma^S(\mathbf{u}_*^\theta)\mathbf{n}^S$ on Γ_0.

Similary, there exists $\mathbf{j}_F \in \left(H_{00}^{1/2}(\Gamma_{\widehat{\mathbf{u}}_*^\theta})\right)'$ defined by

$$\begin{aligned} &\left\langle \mathbf{j}_F, \gamma_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}}(\mathbf{w}^F)\right\rangle_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}} \\ = \;& \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} 2\mu^F \epsilon\left(\mathbf{v}_*^\theta\right) : \epsilon\left(\mathbf{w}^F\right)\, d\mathbf{x} - \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \left(\nabla\cdot\mathbf{w}^F\right) p_*^\theta \, d\mathbf{x} - \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \mathbf{f}^F \cdot \mathbf{w}^F \, d\mathbf{x} \end{aligned} \tag{5.10}$$

for all $\mathbf{w}^F \in \left(H^1(\Omega^F_{\widehat{\mathbf{u}}_*^\theta})\right)^2$, such that $\mathbf{w}^F = 0$ on $\Sigma_1 \cup (\Sigma_2 \setminus \Gamma_D)$, where $\langle\cdot,\cdot\rangle_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}}$ is the duality $\left(H_{00}^{1/2}(\Gamma_{\widehat{\mathbf{u}}_*^\theta})\right)'$, $H_{00}^{1/2}(\Gamma_{\widehat{\mathbf{u}}_*^\theta})$ and $\gamma_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}}$ is the trace on $\Gamma_{\widehat{\mathbf{u}}_*^\theta}$. We can interpret $\mathbf{j}_F$ as $\sigma^F(\mathbf{v}_*^\theta, p_*^\theta)\mathbf{n}^F$ on $\Gamma_{\widehat{\mathbf{u}}_*^\theta}$, see Boyer, Fabrie [5], p. 325.

Proposition 4. *The restrictions of* $\mathbf{v}_*^\theta$ *and* p_*^θ *to* $\Omega^F_{\widehat{\mathbf{u}}_*^\theta}$ *together with* $\mathbf{u}_*^\theta \in \left(H^1(\Omega_0^S)\right)^2$ *verify*

$\nabla_{\mathbf{X}} \cdot \sigma^S\left(\mathbf{u}_*^\theta\right) \in \left(L^2(\Omega_0^S)\right)^2$, $\nabla \cdot \sigma^F\left(\mathbf{v}_*^\theta, p_*^\theta\right)_{|\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \in \left(L^2(\Omega^F_{\widehat{\mathbf{u}}_*^\theta})\right)^2$ *and the following system holds*

$$\begin{aligned}
-\nabla_{\mathbf{X}} \cdot \sigma^S\left(\mathbf{u}_*^\theta\right) &= \mathbf{f}^S, \quad \text{in } \left(L^2(\Omega_0^S)\right)^2 & (5.11)\\
\mathbf{u}_*^\theta &= 0, \quad \text{on } \Gamma_D & (5.12)\\
-\nabla \cdot \sigma^F\left(\mathbf{v}_*^\theta, p_*^\theta\right) &= \mathbf{f}^F, \quad \text{in } \left(L^2(\Omega^F_{\widehat{\mathbf{u}}_*^\theta})\right)^2 & (5.13)\\
\nabla \cdot \mathbf{v}_*^\theta &= 0, \quad \text{in } L^2(\Omega^F_{\widehat{\mathbf{u}}_*^\theta}) & (5.14)\\
\mathbf{v}_*^\theta &= \mathbf{g}, \quad \text{on } \Sigma_1 & (5.15)\\
\mathbf{v}_*^\theta &= 0, \quad \text{on } \Sigma_2 \setminus \Gamma_D & (5.16)\\
\mathbf{v}_*^\theta &= 0, \quad \text{on } \Gamma_{\widehat{\mathbf{u}}_*^\theta} & (5.17)\\
\left\langle \mathbf{j}_F, \gamma_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}}(\mathbf{w}) \right\rangle_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}} &= -\left\langle \mathbf{j}_S, \gamma_{\Gamma_0}(\mathbf{w}_*^S) \right\rangle_{\Gamma_0}. & (5.18)
\end{aligned}$$

The equation (5.18) holds for all $\mathbf{w} \in W$ *and* $\mathbf{w}_*^S = \mathbf{w}_{|\Omega^S_{\widehat{\mathbf{u}}_*^\theta}} \circ \varphi_*^\theta$, *where* $\varphi_*^\theta(\mathbf{X}) = \mathbf{X} + \mathcal{P}_\theta(\mathbf{u}_*^\theta)(\mathbf{X})$. *In (5.18),* $\mathbf{j}_S$ *and* $\mathbf{j}_F$ *are defined by (5.9) and (5.10), respectively. We can interpret (5.18) as the action and reaction principle: the forces acting on the fluid-structure interface are equal in size and opposite in direction.*

Proof. We use the same arguments as in the proof of Proposition 5 from Halanay, Murea, Tiba [22] or Theorem 6.2 from Halanay, Murea, Tiba [24]. When $\varepsilon \to 0$, we have that $\Omega^S_{\widehat{\mathbf{u}}_\varepsilon^\theta} \to \Omega^S_{\widehat{\mathbf{u}}_*^\theta}$ in the complementary Hausdorff-Pompeiu metric. And similarly $\Omega^F_{\widehat{\mathbf{u}}_\varepsilon^\theta} \to \Omega^F_{\widehat{\mathbf{u}}_*^\theta}$ in the same topology.

Since $\mathbf{v}_\varepsilon^\theta = \mathbf{g}$ on Σ_1 and $\mathbf{v}_\varepsilon^\theta = 0$ on $\Sigma_1 \setminus \Gamma_D$ then (5.15) and (5.16) hold by passing to the limit $\varepsilon \to 0$. From (5.5), we have $\mathbf{v}_*^\theta = 0$ a.e. in $\Omega^S_{\widehat{\mathbf{u}}_*^\theta}$ and we obtain (5.17).

Using in (3.7)-(3.8) some test functions $\mathbf{w} \in \left(\mathcal{C}_0^\infty(D)\right)^2$ and $q \in \mathcal{C}_0^\infty(D)$, $\int_D q\, d\mathbf{x} = 0$ with their support in $\Omega^F_{\widehat{\mathbf{u}}_*^\theta}$ and passing to the limit $\varepsilon \to 0$, we get

$$\begin{aligned}
\int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} 2\mu^F \epsilon\left(\mathbf{v}_*^\theta\right) : \epsilon(\mathbf{w})\, d\mathbf{x} - \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} (\nabla \cdot \mathbf{w})\, p_*^\theta\, d\mathbf{x} &= \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x},\\
-\int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \left(\nabla \cdot \mathbf{v}_*^\theta\right) q\, d\mathbf{x} &= 0
\end{aligned}$$

for all $\mathbf{w} \in \left(\mathcal{C}_0^\infty(D)\right)^2$, $supp(\mathbf{w}) \subset \Omega^F_{\widehat{\mathbf{u}}_*^\theta}$ and for all $q \in \mathcal{C}_0^\infty(D)$, $\int_D q\, d\mathbf{x} = 0$, $supp(q) \subset \Omega^F_{\widehat{\mathbf{u}}_*^\theta}$. From the first equality, we get (5.13). We point out that $\int_D p_*^\theta\, d\mathbf{x} = 0$, but $\int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} p_*^\theta\, d\mathbf{x}$ is not necessary zero. Let q' be in $\mathcal{C}_0^\infty\left(\Omega^F_{\widehat{\mathbf{u}}_*^\theta}\right)$ and we set $q = q' - \frac{\int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} q'\, d\mathbf{x}}{\int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} 1\, d\mathbf{x}}$. We have $\int_D q\, d\mathbf{x} = 0$.

Then

$$\begin{aligned}\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} \left(\nabla \cdot \mathbf{v}^\theta_*\right) q'\, d\mathbf{x} &= \int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} \left(\nabla \cdot \mathbf{v}^\theta_*\right) q\, d\mathbf{x} + \frac{\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} q'\, d\mathbf{x}}{\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} 1\, d\mathbf{x}} \int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} \left(\nabla \cdot \mathbf{v}^\theta_*\right)\, d\mathbf{x} \\ &= \frac{\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} q'\, d\mathbf{x}}{\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} 1\, d\mathbf{x}} \int_{\Sigma_1} \mathbf{v}^\theta_* \cdot \mathbf{n}^F\, ds = \frac{\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} q'\, d\mathbf{x}}{\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} 1\, d\mathbf{x}} \int_{\Sigma_1} \mathbf{g} \cdot \mathbf{n}^F\, ds = 0.\end{aligned}$$

We obtain

$$\int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_*}} \left(\nabla \cdot \mathbf{v}^\theta_*\right) q'\, d\mathbf{x} = 0, \quad \forall q' \in \mathcal{C}^\infty_0\left(\Omega^F_{\widehat{\mathbf{u}}^\theta_*}\right).$$

But $\mathcal{C}^\infty_0\left(\Omega^F_{\widehat{\mathbf{u}}^\theta_*}\right)$ is dense in $L^2\left(\Omega^F_{\widehat{\mathbf{u}}^\theta_*}\right)$ and we get (5.14).

From (3.7), with $\mathbf{u} = \mathcal{P}_\theta(\mathbf{u}^\theta_\varepsilon)$, we have

$$\begin{aligned}&\int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} 2\mu^F \epsilon\left(\mathbf{v}^\theta_\varepsilon\right) : \epsilon\left(\mathbf{w}\right)\, d\mathbf{x} - \int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \left(\nabla \cdot \mathbf{w}\right) p^\theta_\varepsilon\, d\mathbf{x} - \int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x} \\ &+\frac{1}{\varepsilon} \int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \left(\mathbf{v}^\theta_\varepsilon \cdot \mathbf{w} + \nabla \mathbf{v}^\theta_\varepsilon : \nabla \mathbf{w}\right) d\mathbf{x} \\ = \; &- \int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_\varepsilon}} 2\mu^F \epsilon\left(\mathbf{v}^\theta_\varepsilon\right) : \epsilon\left(\mathbf{w}\right)\, d\mathbf{x} + \int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \left(\nabla \cdot \mathbf{w}\right) p^\theta_\varepsilon\, d\mathbf{x} \\ &+ \int_{\Omega^F_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x} \end{aligned} \tag{5.19}$$

The left-hand side above is equal to the sum of the last three terms in (3.9) after the change of variable $\varphi^\theta_\varepsilon : \Omega^S_0 \to \Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}$, $\varphi^\theta_\varepsilon\left(\mathbf{X}\right) = \mathbf{X} + \mathcal{P}_\theta(\mathbf{u}^\theta_\varepsilon)\left(\mathbf{X}\right)$. More precisely, we have

$$\begin{aligned}&\int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} 2\mu^F \epsilon\left(\mathbf{v}^\theta_\varepsilon\right) : \epsilon\left(\mathbf{w}\right)\, d\mathbf{x} - \int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \left(\nabla \cdot \mathbf{w}\right) p^\theta_\varepsilon\, d\mathbf{x} - \int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x} \\ &+\frac{1}{\varepsilon} \int_{\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \left(\mathbf{v}^\theta_\varepsilon \cdot \mathbf{w} + \nabla \mathbf{v}^\theta_\varepsilon : \nabla \mathbf{w}\right) d\mathbf{x} \\ = \; &\int_{\Omega^S_0} J\left(\sigma^F\left(\mathbf{v}^\theta_\varepsilon, p^\theta_\varepsilon\right) \circ \varphi^\theta_\varepsilon\right) \mathbf{F}^{-T} : \nabla_{\mathbf{X}} \mathbf{w}^S_\varepsilon\, d\mathbf{X} - \int_{\Omega^S_0} J\left(\mathbf{f}^F \circ \varphi^\theta_\varepsilon\right) \cdot \mathbf{w}^S_\varepsilon\, d\mathbf{X} \\ &+\frac{1}{\varepsilon} \int_{\Omega^S_0} J\left(\left(\mathbf{v}^\theta_\varepsilon \circ \varphi^\theta_\varepsilon\right) \cdot \mathbf{w}^S_\varepsilon + \left(\nabla \mathbf{v}^\theta_\varepsilon \circ \varphi^\theta_\varepsilon\right) \mathbf{F}^{-T} : \nabla_{\mathbf{X}} \mathbf{w}^S_\varepsilon\right)\, d\mathbf{X}\end{aligned} \tag{5.20}$$

where $\mathbf{w}^S_\varepsilon = \mathbf{w}_{|\Omega^S_{\widehat{\mathbf{u}}^\theta_\varepsilon}} \circ \varphi^\theta_\varepsilon$.

From (3.9), (5.19) and (5.20), we obtain

$$
\begin{aligned}
& a_S\left(\mathbf{u}_\varepsilon^\theta, \mathbf{w}_\varepsilon^S\right) - \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}_\varepsilon^S \, d\mathbf{X} \\
= & -\int_{\Omega^F_{\widehat{\mathbf{u}}_\varepsilon^\theta}} 2\mu^F \epsilon\left(\mathbf{v}_\varepsilon^\theta\right) : \epsilon\left(\mathbf{w}\right)\, d\mathbf{x} + \int_{\Omega^F_{\widehat{\mathbf{u}}_\varepsilon^\theta}} \left(\nabla\cdot\mathbf{w}\right) p_\varepsilon^\theta \, d\mathbf{x} \\
& + \int_{\Omega^F_{\widehat{\mathbf{u}}_\varepsilon^\theta}} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x}.
\end{aligned}
\tag{5.21}
$$

For $supp(\mathbf{w}) \subset \Omega^S_{\widehat{\mathbf{u}}_\varepsilon^\theta}$, the right-hand side of the above equation vanishes and we get

$$
a_S\left(\mathbf{u}_\varepsilon^\theta, \mathbf{w}_\varepsilon^S\right) = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}_\varepsilon^S \, d\mathbf{X}
$$

for all $supp(\mathbf{w}_\varepsilon^S) \subset \Omega_0^S$. By passing to the limit $\varepsilon \to 0$ we can obtain that

$$
a_S\left(\mathbf{u}_*^\theta, \mathbf{w}^S\right) = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S \, d\mathbf{X}, \quad \forall \mathbf{w}^S \in \left(\mathcal{C}_0^\infty(\Omega_0^S)\right)^2
$$

which implies (5.11). Also, from $\mathbf{u}_\varepsilon^\theta = 0$ on Γ_D, then (5.12).

It remains to interpret (5.18). By passing to the limit $\varepsilon \to 0$ in (5.21), we get

$$
\begin{aligned}
& a_S\left(\mathbf{u}_*^\theta, \mathbf{w}_*^S\right) - \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}_*^S \, d\mathbf{X} \\
= & -\int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} 2\mu^F \epsilon\left(\mathbf{v}_*^\theta\right) : \epsilon\left(\mathbf{w}\right)\, d\mathbf{x} + \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \left(\nabla\cdot\mathbf{w}\right) p_*^\theta \, d\mathbf{x} \\
& + \int_{\Omega^F_{\widehat{\mathbf{u}}_*^\theta}} \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x}.
\end{aligned}
\tag{5.22}
$$

where $\mathbf{w}_*^S = \mathbf{w}_{|\Omega^S_{\widehat{\mathbf{u}}_*^\theta}} \circ \varphi_*^\theta$ and $\varphi_*^\theta(\mathbf{X}) = \mathbf{X} + \mathcal{P}_\theta(\mathbf{u}_*^\theta)(\mathbf{X})$.

From (5.22), (5.9), (5.10), we get

$$
\left\langle \mathbf{j}_S, \gamma_{\Gamma_0}(\mathbf{w}_*^S)\right\rangle_{\Gamma_0} = -\left\langle \mathbf{j}_F, \gamma_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}}(\mathbf{w})\right\rangle_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}}
$$

for all $\mathbf{w} \in W$ and $\mathbf{w}_*^S = \mathbf{w}_{|\Omega^S_{\widehat{\mathbf{u}}_*^\theta}} \circ \varphi_*^\theta$, which could be interpreted formally by

$$
\int_{\Gamma_0} \left(\sigma^S \mathbf{n}^S\right) \cdot \mathbf{z}\, dS = -\int_{\Gamma_{\widehat{\mathbf{u}}_*^\theta}} \left(\sigma^F \mathbf{n}^F\right) \cdot \left(\mathbf{z} \circ \left(\varphi_*^\theta\right)^{-1}\right)\, ds, \quad \forall \mathbf{z} \in \left(L^2(\Gamma_0)\right)^2.
$$

□

Remark 4. *The system (5.11)-(5.18) is similar to (2.1)-(2.8). The unknown* $\mathbf{u}_*^\theta$ *appears in (5.11), (5.12), while in (5.13), (5.14), (5.17), (5.18), the fluid domain depends on* $\widehat{\mathbf{u}}_*^\theta = \mathcal{P}_\theta(\mathbf{u}_*^\theta)$.

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