Simulation numérique d'une problème d'interaction entre un fluide pulsatif et une structure élastique

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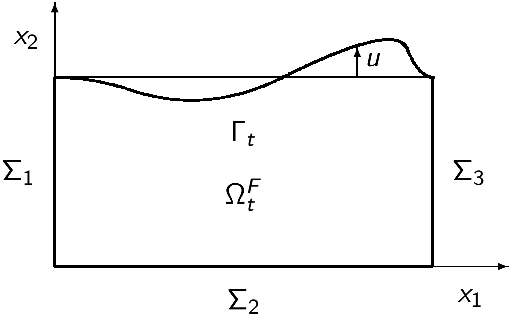
- Approximation of the coupled fluid-structure equations. Identification of the stresses on the interface using the Least Squares Method.
- Numerical results

Example of an admissible domain



Strong equations of the structure

Find the transverse displacement $u : [0, L] \times [0, T] \to \mathbb{R}$ such that

$$
\begin{aligned}
&\rho^S h^S \frac{\partial^2 u}{\partial t^2}(x_1, t) + \frac{E(h^S)^3}{12(1-\nu^2)} \frac{\partial^4 u}{\partial x_1^4}(x_1, t) = \eta(x_1, t),\\
&u(0, t) = 0, \ \frac{\partial u}{\partial x_1}(0, t) = 0, \quad t \in (0, T)\\
&u(L, t) = 0, \ \frac{\partial u}{\partial x_1}(L, t) = 0, \quad t \in (0, T)\\
&u(x_1, 0) = u^0(x_1), \quad x_1 \in (0, L)\\
&\frac{\partial u}{\partial t}(x_1, 0) = \dot{u}^0(x_1), \quad x_1 \in (0, L)
\end{aligned}
$$

Strong form of the unsteady Navier-Stokes equations

$$\begin{aligned}
\rho^F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right) - \mu \Delta \mathbf{v} + \nabla p &= \mathbf{f}^F, \quad \forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F \\
\nabla \cdot \mathbf{v} &= 0, \quad \forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_1 \times (0, T) \\
p &= P_{in}, \quad \text{on } \Sigma_1 \times (0, T) \\
\mathbf{v} &= \mathbf{g}, \quad \text{on } \Sigma_2 \times (0, T) \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_3 \times (0, T) \\
p &= P_{out}, \quad \text{on } \Sigma_3 \times (0, T) \\
\mathbf{v}\left(x_1, H + u(x_1, t), t\right) &= \left(0, \frac{\partial u}{\partial t}(x_1, t) \right)^T, \\
& \quad \forall (x_1, t) \in (0, L) \times (0, T) \\
\mathbf{v}(\mathbf{x}, 0) &= \mathbf{v}^0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_0^F
\end{aligned}$$

Strong form of the coupled equations

Find the transverse displacement u of the structure, the velocity $\mathbf{v}$ and the pressure p of the fluid such that

$$\eta(x_1, t) = -\left(\sigma^F \mathbf{n} \cdot \mathbf{e}_2\right)_{(x_1, H+u(x_1,t))} \sqrt{1 + \left(\frac{\partial u}{\partial x_1}(x_1, t)\right)^2}$$

where $\sigma^F = -p\, I + \mu\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right)$ is the stress tensor of the fluid, $\mathbf{e}_2 = (0,1)^T$ is the unit vector in the x_2 direction.
The displacement of the structure depends on the vertical component of the stresses exerced by the fluid on the interface. The movement of the structure changes the domain where the fluid equations must be solved. Also, on the interface we have to impose the equality between the fluid and structure velocity.

Natural frequencies and normal mode shapes

For each $i \in \mathbb{N}$ there exists an unique normal mode shape $\phi_i \in \mathcal{C}^4([0, L])$ such that

$$\begin{aligned}
&\phi_i''''(x_1) = (a_i)^4 \phi_i(x_1), \quad x_1 \in (0, L) \\
&\phi_i(0) = \frac{\partial \phi_i}{\partial x_1}(0) = 0, \\
&\phi_i(L) = \frac{\partial \phi_i}{\partial x_1}(L) = 0, \\
&\int_0^L \phi_i^2(x_1)\, dx_1 = 1.
\end{aligned}$$

The normal mode shapes ϕ_i for $i \in \mathbb{N}$ form an orthogonal basis of $L^2(0, L)$.

$$\eta(x_1, t) = \sum_{i \geq 0} \alpha_i(t)\phi_i(x_1), \quad u(x_1, t) = \sum_{i \geq 0} q_i(t)\phi_i(x_1)$$

where q_i is the solution of the second order differential equation

$$\begin{aligned}
q_i''(t) + \omega_i^2 q_i(t) &= \frac{1}{\rho^S h^S}\alpha_i(t), \quad t \in (0, T) \\
q_i(0) &= \int_0^L u^0(x_1)\phi_i(x_1)\, dx_1 \\
q_i'(0) &= \int_0^L \dot{u}^0(x_1)\phi_i(x_1)\, dx_1.
\end{aligned}$$

Newmark method

Knowing q_i^n, $\dot{q}_i^n$, $\ddot{q}_i^n$ and α_i^{n+1}, find q_i^{n+1}, $\dot{q}_i^{n+1}$, $\ddot{q}_i^{n+1}$ such that:

$$\begin{aligned}
&\ddot{q}_i^{n+1} + \omega_i^2 q_i^{n+1} = \frac{1}{\rho^S h^S}\alpha_i^{n+1},\\
&\dot{q}_i^{n+1} = \dot{q}_i^n + \Delta t\left[(1-\delta)\ddot{q}_i^n + \delta\ddot{q}_i^{n+1}\right],\\
&q_i^{n+1} = q_i^n + \Delta t\dot{q}_i^n + (\Delta t)^2\left[\left(\frac{1}{2}-\theta\right)\ddot{q}_i^n + \theta\ddot{q}_i^{n+1}\right]
\end{aligned}$$

where δ and θ are two real parameters.
This method is unconditional stable for $2\theta \geq \delta \geq 1/2$. It is first order accuracy if $\delta \neq 1/2$. If $\delta = 1/2$, it is second order accuracy in the case $\theta \neq 1/12$ and forth order accuracy is achieved if $\theta = 1/12$.

Arbitrary Lagrangian Eulerian (ALE) coordinates

$$\widehat{\Omega}^F = (0, L) \times (0, H), \quad \widehat{\Gamma} = (0, L) \times \{H\}$$

$$\mathcal{A}_t\left(\widehat{x}_1, \widehat{x}_2\right) = \left(\widehat{x}_1, \frac{H + u\left(\widehat{x}_1, t\right)}{H}\widehat{x}_2\right)^T$$

$$\mathbf{x} = \mathcal{A}_t\left(\widehat{\mathbf{x}}\right), \qquad \widehat{\mathbf{v}}(\widehat{\mathbf{x}}, t) = \mathbf{v}\left(\mathcal{A}_t(\widehat{\mathbf{x}}), t\right)$$

$$\frac{\partial \mathbf{v}}{\partial t}\left(\mathbf{x}, t\right) = \frac{\partial \widehat{\mathbf{v}}}{\partial t}(\widehat{\mathbf{x}}, t) - \left(\frac{\partial \mathcal{A}_t}{\partial t}(\widehat{\mathbf{x}}) \cdot \nabla\right)\mathbf{v}\left(\mathbf{x}, t\right)$$

$$\frac{D^{\mathcal{A}}\mathbf{v}}{Dt}\left(\mathbf{x}, t\right) \stackrel{def}{=} \frac{\partial \widehat{\mathbf{v}}}{\partial t}(\widehat{\mathbf{x}}, t)$$

Approximation of the ALE derivative

$$\frac{\partial \widehat{\mathbf{v}}}{\partial t}(\widehat{\mathbf{x}}, t_{n+1}) \approx \frac{\widehat{\mathbf{v}}(\widehat{\mathbf{x}}, t_{n+1}) - \widehat{\mathbf{v}}(\widehat{\mathbf{x}}, t_n)}{\Delta t} = \frac{\mathbf{v}\left(\mathcal{A}_{t_{n+1}}(\widehat{\mathbf{x}}), t_{n+1}\right) - \mathbf{v}(\mathcal{A}_{t_n}(\widehat{\mathbf{x}}), t_n)}{\Delta t}$$
$$= \frac{\mathbf{v}(\mathbf{x}, t_{n+1}) - \mathbf{v}\left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x}), t_n\right)}{\Delta t} \approx \frac{\mathbf{v}^{n+1}(\mathbf{x}) - \mathbf{v}^n\left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x})\right)}{\Delta t}.$$

$$\boldsymbol{\vartheta}^{n+1}(\mathbf{x}) \stackrel{def}{=} \frac{\partial \mathcal{A}_t}{\partial t}(\widehat{\mathbf{x}})|_{t=t_{n+1}} = \left(0, \frac{\partial u}{\partial t}(x_1, t_{n+1}) \frac{x_2}{H + u(x_1, t_{n+1})}\right)^T$$

$$\frac{\partial \mathbf{v}}{\partial t}(\mathbf{x}, t_{n+1}) \approx \frac{\mathbf{v}^{n+1}(\mathbf{x}) - \mathbf{v}^n\left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x})\right)}{\Delta t} - \left(\boldsymbol{\vartheta}^{n+1}(\mathbf{x}) \cdot \nabla\right) \mathbf{v}^{n+1}(\mathbf{x})$$

$$\mathbf{V}^n(\mathbf{x}) = \mathbf{v}^n\left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x})\right)$$

$$
\begin{aligned}
\rho^F \left(\frac{\mathbf{v}^{n+1}}{\Delta t} + \left(\left(\mathbf{V}^n - \boldsymbol{\vartheta}^{n+1} \right) \cdot \nabla \right) \mathbf{v}^{n+1} \right) & \\
-\mu \Delta \mathbf{v}^{n+1} + \nabla p^{n+1} &= \rho^F \frac{\mathbf{V}^n}{\Delta t} + \mathbf{f}^F \\
& \quad \text{in } \Omega^F_{t_{n+1}} \\
\nabla \cdot \mathbf{v}^{n+1} &= 0 \text{ in } \Omega^F_{t_{n+1}} \\
\mathbf{v}^{n+1} \times \mathbf{n} &= 0 \text{ on } \Sigma_1 \\
p^{n+1} &= P_{in}(\cdot, t_{n+1}) \text{ on } \Sigma_1 \\
\mathbf{v}^{n+1} &= \mathbf{g}(\cdot, t_{n+1}) \text{ on } \Sigma_2 \\
\mathbf{v}^{n+1} \times \mathbf{n} &= 0 \text{ on } \Sigma_3 \\
p^{n+1} &= P_{out}(\cdot, t_{n+1}) \text{ on } \Sigma_3 \\
\mathbf{v}^{n+1} \left(x_1, H + u(x_1, t_{n+1}), t \right) &= \left(0, \frac{\partial u}{\partial t}(x_1, t_{n+1}) \right)^T, \\
& \quad 0 < x_1 < L.
\end{aligned}
$$

Mixed Finite Element

$$
\begin{aligned}
W^{n+1} &= \Big\{ \mathbf{w} \in \left(H^1 \left(\Omega^F_{t_{n+1}} \right) \right)^2 ; \\
&\quad \mathbf{w} \times \mathbf{n} = 0 \text{ on } \Sigma_1 \cup \Sigma_3, \ \ \mathbf{w} = 0 \text{ on } \Sigma_2 \cup \Gamma_{t_{n+1}} \Big\}, \\
Q^{n+1} &= L^2 \left(\Omega^F_{t_{n+1}} \right).
\end{aligned}
$$

Find the velocity $\mathbf{v}^{n+1}$ and the pressure p^{n+1} such that

$$
\left\{ \begin{array}{lcl}
a_F^{n+1}\left(\mathbf{v}^{n+1}, \mathbf{w}\right) + d_F^{n+1}\left(\mathbf{v}^{n+1}, \mathbf{w}\right) + b_F^{n+1}\left(\mathbf{w}, p^{n+1}\right) & = & \ell^{n+1}\left(\mathbf{w}\right), \forall \mathbf{w} \\
b_F^{n+1}\left(\mathbf{v}^{n+1}, q\right) & = & 0, \forall q
\end{array} \right.
$$

$$
\begin{aligned}
a_F^{n+1}\left(\mathbf{v}^{n+1},\mathbf{w}\right) &= \frac{\rho^F}{\Delta t}\left(\mathbf{v}^{n+1},\mathbf{w}\right) \\
&\quad +\mu\left(\nabla\times\mathbf{v}^{n+1},\nabla\times\mathbf{w}\right)+\mu\left(\nabla\cdot\mathbf{v}^{n+1},\nabla\cdot\mathbf{w}\right) \\
d_F^{n+1}\left(\mathbf{v}^{n+1},\mathbf{w}\right) &= \rho^F\left(\left(\left(\mathbf{V}^n-\boldsymbol{\vartheta}^{n+1}\right)\cdot\nabla\right)\mathbf{v}^{n+1},\mathbf{w}\right) \\
b_F^{n+1}\left(\mathbf{w},q\right) &= -\left(\nabla\cdot\mathbf{w},q\right) \\
\ell^{n+1}\left(\mathbf{w}\right) &= \frac{\rho^F}{\Delta t}\left(\mathbf{V}^n,\mathbf{w}\right)+\left(\mathbf{f}^F,\mathbf{w}\right) \\
&\quad -\int_{\Sigma_1} P_{in}(\cdot,t_{n+1})\mathbf{n}\cdot\mathbf{w}\,d\gamma \\
&\quad -\int_{\Sigma_3} P_{out}(\cdot,t_{n+1})\mathbf{n}\cdot\mathbf{w}\,d\gamma
\end{aligned}
$$

Weak form of time derivative

Time derivative inside the integral

$$\int_{\Omega_t^F} \frac{D^{\mathcal{A}}\mathbf{v}}{Dt}(\mathbf{x},t)\cdot\mathbf{w}(\mathbf{x},t)\,d\mathbf{x}, \quad \forall \mathbf{w}(\cdot,t) \in \left(H^1\left(\Omega_t^F\right)\right)^2$$

Time derivative outside the integral

$$= \frac{d}{dt}\int_{\Omega_t^F} \mathbf{v}(\mathbf{x},t)\cdot\mathbf{w}(\mathbf{x},t)\,d\mathbf{x} - \int_{\Omega_t^F} \mathbf{v}(\mathbf{x},t)\cdot\mathbf{w}(\mathbf{x},t)\,\nabla\cdot\boldsymbol{\vartheta}(\mathbf{x},t)\,d\mathbf{x},$$

forall $\mathbf{w}$ such that $\frac{\partial \widehat{\mathbf{w}}}{\partial t}(\widehat{\mathbf{x}},t) = 0$.
For example, we can take $\mathbf{w}(\mathbf{x},t) = \widehat{\mathbf{w}}\left(\mathcal{A}_t^{-1}(\mathbf{x})\right)$, where $\widehat{\mathbf{w}} \in \left(H^1\left(\widehat{\Omega}^F\right)\right)^2$.

Strategies for solving at each time step the coupled problem

The fixed point and the root finding frameworks:

$$\mathcal{F} \circ \mathcal{S}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}, \qquad \mathcal{F} \circ \mathcal{S}(\boldsymbol{\alpha}) - \boldsymbol{\alpha} = 0.$$

Fixed point (with eventually a relaxation parameter):
- Nobile 2001, Formaggia et al 2001

Block Newton:
- Steindorf and Matthies 2002, the derivative are approached by finite differences
- Gerbeau, Vidrascu 2003, the tangent operator is approached
- Fernandez and Moubachir 2004, the derivative was computed exactly

The optimization approach

If the starting point is not chosen “sufficiently close” to the solution, fixed point or Newton like methods diverge.

The continuity of the stresses on the interface will be treated by the Least Square Method and at each time step we have to solve an optimization problem which is less sensitive to the choice of the starting point. This is the main advantage of this approach.
Our approach is to minimize

$$J^{n+1}(\boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_2^2 = \frac{1}{2}\|\boldsymbol{\alpha} - \mathcal{F} \circ \mathcal{S}(\boldsymbol{\alpha})\|_2^2$$

Identification of the stresses on the interface using the Least Squares Method

The stresses on the interface at the current time step t_{n+1} will be approached by $\eta_m^{n+1}(x_1) = \sum_{i=0}^{m-1} \alpha_i^{n+1}\phi_i(x_1)$.
The parameters α_i^{n+1} for $0 \leq i \leq m-1$ will be "identified" solving an optimization problem

$$\boldsymbol{\alpha}^{n+1} \stackrel{def}{=} \left(\alpha_0^{n+1}, \ldots, \alpha_{m-1}^{n+1}\right) \in \arg\min_{\boldsymbol{\alpha}\in\mathbb{R}^m} J^{n+1}(\boldsymbol{\alpha}).$$

Structure sub-problem
Knowing q_i^n, $\dot{q}_i^n$, $\ddot{q}_i^n$, find Q_i, $\dot{Q}_i$, $\ddot{Q}_i$ by Newmark method.
Set

$$U(x_1) = \sum_{i=0}^{m-1} Q_i\phi_i(x_1),\ \dot{U}(x_1) = \sum_{i=0}^{m-1} \dot{Q}_i\phi_i(x_1),\ \ddot{U}(x_1) = \sum_{i=0}^{m-1} \ddot{Q}_i\phi_i(x_1).$$

Mesh construction

Let $\widehat{\mathcal{T}}_h$ be a mesh with triangular elements of the reference domain $\widehat{\Omega}^F$.

We define the mesh with triangular elements $\mathcal{T}_h$ by moving each node of $\widehat{\mathcal{T}}_h$ using the map

$$\mathcal{A}_U(\widehat{x}_1, \widehat{x}_2) = \left(\widehat{x}_1, \frac{H + U(\widehat{x}_1)}{H}\widehat{x}_2\right)^T.$$

Fluid sub-problem

$$
\begin{aligned}
W_h &= \left\{ \mathbf{w}_h \in \left(C^0\left(\overline{\Omega}_h^F\right)\right)^2 ;\ \forall K \in \mathcal{T}_h,\ \mathbf{w}_h|_K \in P1 + bubble, \right. \\
&\qquad \left. \mathbf{w}_h \times \mathbf{n} = 0 \text{ on } \Sigma_1 \cup \Sigma_3,\ \mathbf{w}_h = 0 \text{ on } \Sigma_2 \cup \Gamma_h \right\}, \\
Q_h &= \left\{ q_h \in C^0\left(\overline{\Omega}_h^F\right) ;\ \forall K \in \mathcal{T}_h,\ q_h|_K \in P1 \right\}.
\end{aligned}
$$

Find the velocity $\mathbf{v}_h$ satisfies the boundary conditions

$$
\begin{aligned}
\mathbf{v}_h \times \mathbf{n} &= 0, \text{ on each vertex of } \Sigma_1 \cup \Sigma_3, \\
\mathbf{v}_h &= \mathbf{g}(\cdot, t_{n+1}), \text{ on each vertex of } \Sigma_2, \\
\mathbf{v}_h &= \left(0, \dot{U}\right)^T, \text{ on each vertex of the top boundary } \Gamma_h
\end{aligned}
$$

and the pressure $p_h \in Q_h$ such that

$$
\left\{
\begin{array}{lcl}
a_F^{n+1}(\mathbf{v}_h, \mathbf{w}_h) + d_F^{n+1}(\mathbf{v}_h, \mathbf{w}_h) + b_F^{n+1}(\mathbf{w}_h, p_h) & = & \ell^{n+1}(\mathbf{w}_h) \\
b_F^{n+1}(\mathbf{v}_h, q_h) & = & 0
\end{array}
\right.
$$

Definition of the cost function

$$\beta_i = -\int_0^L \phi_i(x_1) \left(\sigma^F \left(\mathbf{v}_h, p_h\right) \mathbf{n} \cdot \mathbf{e}_2\right)_{(x_1, H+U(x_1))} \sqrt{1 + \left(\frac{\partial U}{\partial x_1}(x_1)\right)^2}$$

$$\beta_i = \int_0^L \phi_i(x_1) \left(p_h + \mu \left(\frac{\partial v_{h,1}}{\partial x_2} + \frac{\partial v_{h,2}}{\partial x_1}\right) \frac{\partial U}{\partial x_1} - 2\mu \frac{\partial v_{h,2}}{\partial x_2}\right)_{(x_1, H+U(x_1))}$$

Set the cost function

$$J^{n+1}(\boldsymbol{\alpha}) = \frac{1}{2} \sum_{i=0}^{m-1} \left(\alpha_i - \beta_i\right)^2.$$

$$\beta_i = \int_0^L \phi_i(x_1) p_h \left(x_1, H + U(x_1)\right) dx_1, \quad i = 0, \dots, m-1.$$

BFGS scheme

Step 0 Choose a starting point $\alpha^0 \in \mathbb{R}^m$, an $m \times m$ symmetric positive matrix H_0 and a positive scalar ϵ. Set $k = 0$.
Step 1 Compute $\nabla J(\alpha^k)$.
Step 2 If $\left\|\nabla J(\alpha^k)\right\| < \epsilon$ stop.
Step 3 Set $\mathbf{d}^k = -H_k \nabla J(\alpha^k)$.
Step 4 Determine $\alpha^{k+1} = \alpha^k + \theta_k \mathbf{d}^k$, $\theta_k > 0$ by means of an approximate minimization

$$J(\alpha^{k+1}) \approx \min_{\theta \geq 0} J(\alpha^k + \theta \mathbf{d}^k).$$

Step 5 Compute $\delta_k = \alpha^{k+1} - \alpha^k$.
Step 6 Compute $\nabla J(\alpha^{k+1})$ and $\gamma_k = \nabla J(\alpha^{k+1}) - \nabla J(\alpha^k)$.
Step 7 Compute

$$H_{k+1} = H_k + \left(1 + \frac{\gamma_k^T H_k \gamma_k}{\delta_k^T \gamma_k}\right) \frac{\delta_k \delta_k^T}{\delta_k^T \gamma_k} - \frac{\delta_k \gamma_k^T H_k + H_k \gamma_k \delta_k^T}{\delta_k^T \gamma_k}$$

Step 8 Update $k = k + 1$ and go to the **Step 2**.

The matrices H_k approach the inverse of the hessian of J.

For the inaccurate line search at the **Step 4**, the methods of Goldstein and Armijo were used.

We compute $\nabla J(\alpha)$ by the Finite Differences Method

$$\frac{\partial J}{\partial \alpha_k}(\alpha) \approx \frac{J(\alpha + \Delta\alpha_k \mathbf{e_k}) - J(\alpha)}{\Delta\alpha_k}$$

where $\mathbf{e_k}$ is the k-th vector of the canonical base of $\mathbb{R}^m$ and $\Delta\alpha_k > 0$ is the grid spacing.

Fixed point, Newton and BFGS Methods

$$G : \mathbb{R}^m \to \mathbb{R}^m, \quad G(\boldsymbol{\alpha}) \stackrel{def}{=} \mathcal{F} \circ \mathcal{S}(\boldsymbol{\alpha})$$

- Fixed point iterations

$$G(\boldsymbol{\alpha}) = \boldsymbol{\alpha}, \quad \boldsymbol{\alpha}^{k+1} = G(\boldsymbol{\alpha}^k)$$

- Newton method

$$F(\boldsymbol{\alpha}) \stackrel{def}{=} \boldsymbol{\alpha} - G(\boldsymbol{\alpha}) = 0, \quad \boldsymbol{\alpha}^{k+1} = \boldsymbol{\alpha}^k - \left(\nabla F(\boldsymbol{\alpha}^k)^T\right)^{-1} F(\boldsymbol{\alpha}^k)$$

- BFGS method

$$\inf_{\boldsymbol{\alpha} \in \mathbb{R}^m} J(\boldsymbol{\alpha}) = \frac{1}{2} \|F(\boldsymbol{\alpha})\|^2, \quad \nabla J(\boldsymbol{\alpha}) = (\nabla F(\boldsymbol{\alpha})) F(\boldsymbol{\alpha})$$

Local minimizer and zero residual

If $\boldsymbol{\alpha}^*$ is a local minimizer, then $(\nabla F(\boldsymbol{\alpha}^*))\, F(\boldsymbol{\alpha}^*) = 0$.

What is most surprising is the fact that if the Jacobian matrix $\nabla F(\boldsymbol{\alpha}^*)^T$ is nonsingular, we obtain that $F(\boldsymbol{\alpha}^*) = 0$!

In other words, a local minimizer $\boldsymbol{\alpha}^*$, with nonsingular Jacobian matrix $\nabla F(\boldsymbol{\alpha}^*)^T$, is a global minimizer of zero residual, i.e. $J(\boldsymbol{\alpha}^*) = 0$.

Only in the case when $\nabla F(\boldsymbol{\alpha}^*)^T$ is singular and $F(\boldsymbol{\alpha}^*) \neq 0$, the solution computed by the BFGS Method is not a solution of the fluid-structure coupled problem.

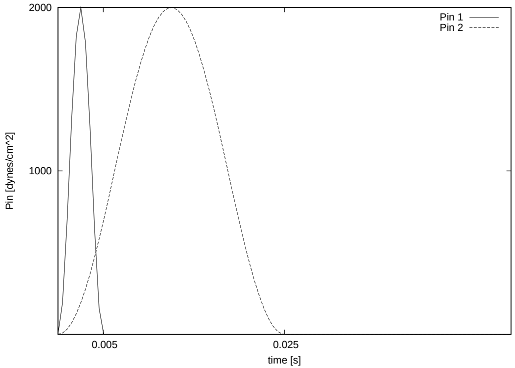
We have to recall that the Newton method fails if $\nabla F(\boldsymbol{\alpha}^*)^T$ is nonsingular.

Numerical results

- The computation has been made in a domain of length $L = 6\ cm$ and height $H = 1\ cm$.
- The viscosity of the blood was taken to be $\mu = 0.035\ \frac{g}{cm \cdot s}$, its density $\rho^F = 1\ \frac{g}{cm^3}$.
- The thickness of the vessel is $h = 0.1\ cm$, the Poisson ratio $\nu = 0.5$, the density $\rho^S = 1.1\ \frac{g}{cm^3}$.
- The number of the normal mode shapes is $m = 5$.
- The gradient of the cost function was approached by the Finite Difference Method with the grid spacing $\Delta\alpha_k = 0.001$.

The numerical tests have been produced using *FreeFem++*.

Input pressure



1) Young modulus $E = 0.75 \cdot 10^6 \ \frac{g}{cm \cdot s^2}$, Final time $T = 0.25\ s$
2) Young modulus $E = 3 \cdot 10^6 \ \frac{g}{cm \cdot s^2}$, Final time $T = 0.1\ s$

Case of an impulsive pressure wave in a higher compliant channel

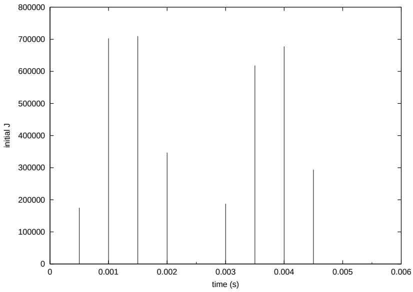
For the boundary conditions we have used:

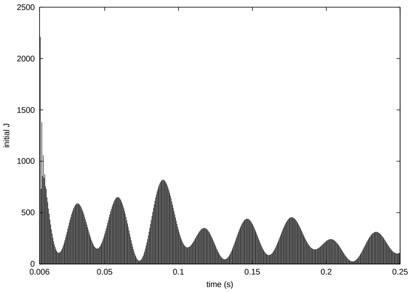
$$
\begin{aligned}
P_{in}(\mathbf{x}, t) &= \begin{cases} 10^3(1-\cos(2\pi t/0.005)), & \mathbf{x} \in \Sigma_1, 0 \le t \le 0.005 \\ 0, & \mathbf{x} \in \Sigma_1, 0.005 \le t \le T \end{cases} \\
\mathbf{g}(\mathbf{x}, t) &= 0, \quad \mathbf{x} \in \Sigma_2, 0 \le t \le T \\
P_{out}(\mathbf{x}, t) &= 0, \quad \mathbf{x} \in \Sigma_3, 0 \le t \le T
\end{aligned}
$$

Δt	mesh size h	no. triangles	no. vertices
0.0005	$h_1 = 0.25$	196	127
0.0005	$h_2 = 0.17$	226	448
0.0005	$h_3 = 0.10$	1250	696

We have performed the simulation for $N = 500$ time steps.
At each time step, we have performed 8 iterations of the BFGS algorithm and 4 iterations in the method for the line search.

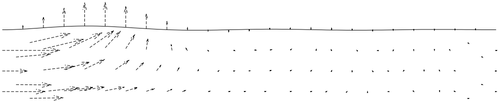
Starting values of the cost function during the pressure impulse (at the left) and after (at the right)



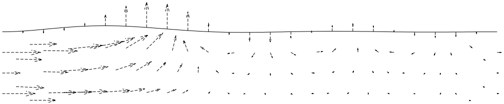


Displacements of the top wall and fluid velocity

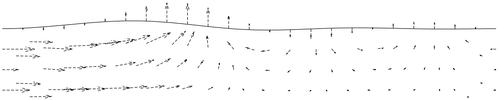
t= 0.0150



t= 0.0300



t= 0.0450



Case of a sine wave of the pressure input in a less compliant vessel

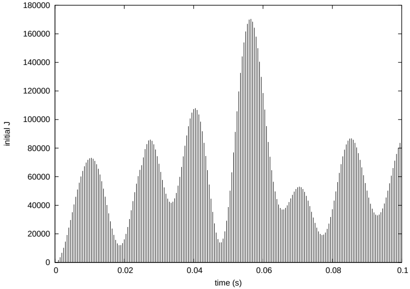
The Young modulus: $E = 3 \cdot 10^6 \; \frac{g}{cm \cdot s^2}$.
The pressure at the inflow:

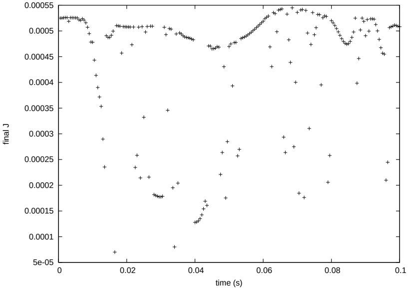
$$P_{in}(\mathbf{x}, t) = \begin{cases} 10^3(1 - \cos(2\pi t/0.025)), & \mathbf{x} \in \Sigma_1, 0 \leq t \leq 0.025 \\ 0, & \mathbf{x} \in \Sigma_1, 0.025 \leq t \leq T \end{cases}$$

Δt	h	N	T
0.0005	0.17	200	0.1
0.0010	0.17	100	0.1
0.0025	0.17	40	0.1

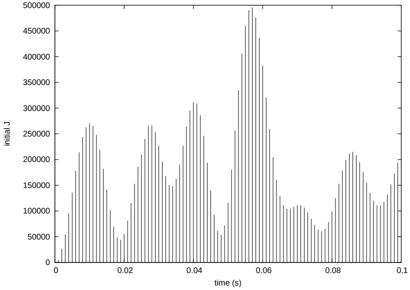
We have performed 10 iterations of the BFGS algorithm and 5 iterations in the method for the line search.

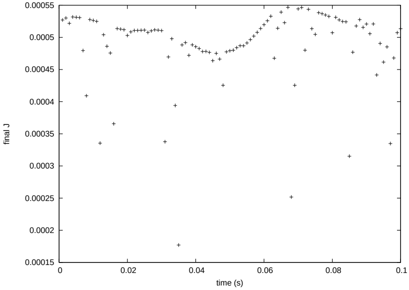
Starting (left) and final (right) values of the cost function for $\Delta t = 0.0005$



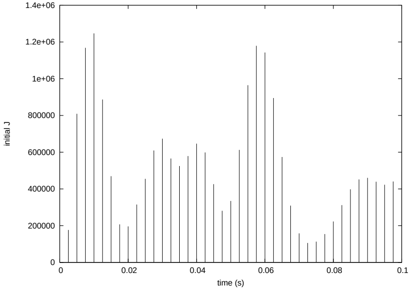


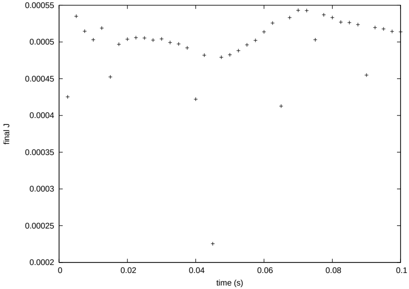
Starting (left) and final (right) values of the cost function for $\Delta t = 0.0010$



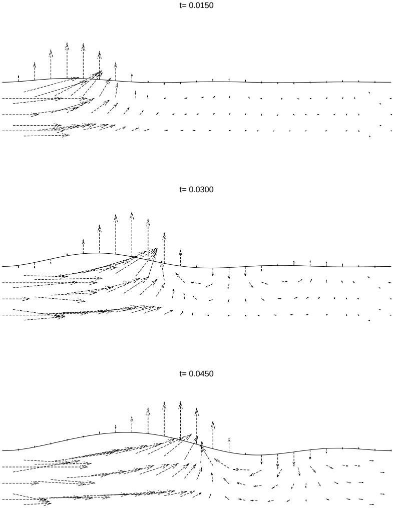


Starting (left) and final (right) values of the cost function for $\Delta t = 0.0025$





Displacements of the top wall and fluid velocity



Conclusions

- The continuity of the stresses at the interface was treated by the Least Squares Method and at each time step we have to solve an optimization problem which is less sensitive to the choice of the starting point and it permits us to use moderate time step. This is the main advantage of this approach.
- In order to solve the optimization problem, we have employed the BFGS method which is successful from farther starting point. The gradient of the cost function was approached by the Finite Difference Method.
- The coupled fluid-structure algorithm has good stability properties.