Numerical method for a two dimensional steady incompressible fluid - beam interaction

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Beam equations. Strong formulation

Let $\lambda_0 : (0, L) \to \mathbb{R}$ be a given continuous function, such that

$$\int_0^L \lambda_0(x_1)\, dx_1 = 0.$$

Find $u : [0, L] \to \mathbb{R}$ the displacement of the beam and $c \in \mathbb{R}$, such that

$$u''''(x_1) = \frac{1}{D}\left(f^S(x_1) + \lambda_0(x_1) + c\right), \quad \forall x_1 \in (0, L)$$
$$u(0) = u(L) = u'(0) = u'(L) = 0$$
$$\int_0^L u(x_1)\, dx_1 = 0$$

The fluid force acting on the beam is $\lambda_0(x_1) + c$.

We impose

$$0 < e \leq H + u(x_1), \quad \forall x_1 \in (0, L)$$

in order to have a connected domain for the fluid.

Fluid equations. Strong formulation

Let $g = (g_1, g_2)^T : \Sigma \to \mathbb{R}^2$ be the imposed velocity profile of the fluid on the rigid boundary, such that

$$\int_\Sigma g \cdot n \, d\sigma = 0.$$

Find

- $v = (v_1, v_2)^T : \Omega_u^F \to \mathbb{R}^2$ the velocity and
- $p : \Omega_u^F \to \mathbb{R}$ the pressure of the fluid,

such that

$$\begin{aligned} -\mu \Delta v + \nabla p &= f^F, \quad in\ \Omega_u^F \\ \text{div}\ v &= 0, \quad in\ \Omega_u^F \\ v &= g, \quad on\ \Sigma \\ v &= 0, \quad on\ \Gamma_u \end{aligned}$$

We will take p_0 such that

$$\int_0^L p_0\,(x_1, H + u(x_1))\ dx_1 = 0.$$

For the Newtonian fluids, the stress tensor σ has the form

$$\sigma = -p\, I + \mu \left(\nabla v + \nabla v^T \right),$$

consequently, the fluid forces acting on the structure are $-\sigma n$.

Proposition If $v \in \left(H^2 \left(\Omega_u^F \right) \right)^2$, $p \in H^1 \left(\Omega_u^F \right)$, v is constant on Γ_u, div $v = 0$ in Ω_u^F, then $-(\sigma n) \cdot n = p$ on Γ_u.

Under the assumption of small displacement of the beam, it follows that $n \approx (0, 1)^T$. Then, it is reasonable to solve the beam equation under the fluid forces given by $p\,(x_1, H + u(x_1))$, $x_1 \in (0, L)$.

Position of the problem

The problem is to find:
$u : [0, L] \to \mathbb{R}$ the displacement of the structure,
$v = (v_1, v_2)^T : \Omega_u^F \to \mathbb{R}^2$ the velocity and
$p : \Omega_u^F \to \mathbb{R}$ the pressure of the fluid, such that

$$
\begin{aligned}
u''''(x_1) &= \tfrac{1}{D}\left(f^S(x_1) + p\left(x_1, H + u(x_1)\right)\right) \\
u(0) &= u(L) = u'(0) = u'(L) = 0 \\
\int_0^L u(x_1)\, dx_1 &= 0 \\
e &\leq H + u(x_1), \quad \forall x_1 \in (0, L) \\
-\mu \Delta v + \nabla p &= f^F, \quad in\ \Omega_u^F \\
\text{div } v &= 0, \quad in\ \Omega_u^F \\
v &= g, \quad on\ \Sigma \\
v &= 0, \quad on\ \Gamma_u
\end{aligned}
$$

Fixed point problem

For a given function $\lambda_0 : (0, L) \to \mathbb{R}$, such that

$$\int_0^L \lambda_0(x_1)\, dx_1 = 0,$$

we can solve the beam equations

$$u''''(x_1) = \frac{1}{D}\left(f^S(x_1) + \lambda_0(x_1) + c\right), \quad \forall x_1 \in (0, L)$$
$$u(0) = u(L) = u'(0) = u'(L) = 0$$
$$\int_0^L u(x_1)\, dx_1 = 0$$

where c is a real constant. If

$$H + u(x_1) \geq e, \, \forall x_1 \in [0, L]$$

we can solve the Stokes equations and we obtain v and p.

We take p_0 such that $\int_0^L p_0(x_1, H + u(x_1))\, dx_1 = 0$.

The fixed point approach is to find λ_0 such that

$$\lambda_0(x_1) = p_0(x_1, H + u(x_1)), \ \forall x_1 \in (0, L).$$

Optimal control

Let $\phi_i : [0, L] \to \mathbb{R}$ be some particular given functions such that $\int_0^L \phi_i(x_1)dx_1 = 0$ and α_i are the scalar parameters to be identified, $1 \leq i \leq p$. We set $\alpha = (\alpha_1, \ldots, \alpha_p)$.

$$\inf J(\alpha) = \int_0^L \left(\sum_{i=1}^{p} \alpha_i \phi_i(x_1) - p_0(x_1, H + u(x_1)) \right)^2$$

$$u''''(x_1) = \frac{1}{D} \left(f^S(x_1) + \sum_{i=1}^{p} \alpha_i \phi_i(x_1) + c(\alpha) \right)$$

$$u(0) = u(L) = u'(0) = u'(L) = 0$$

$$\int_0^L u(x_1)\, dx_1 = 0$$

$$H + u(x_1) \geq e$$

$$-\mu \Delta v + \nabla p = f^F, \quad in\ \Omega_u^F$$

$$\text{div } v = 0, \quad in\ \Omega_u^F$$

$$v = g, \quad on\ \Sigma$$

$$v = 0, \quad on\ \Gamma_u$$

$$p_0 = p - \frac{1}{L} \int_0^L p(x_1, H + u(x_1))\, dx_1$$

Numerical results

We are interested in simulating the blood flow through medium vessels.

Vessel: $L = 3\ cm,\ H = 0.5\ cm,\ h = 0.1\ cm,$ $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2},\ \rho^S = 1.1\ \frac{g}{cm^3},\ g_0 = 981\ \frac{cm}{s^2},$ $f^S(x_1) = -g_0 \rho^S h.$

Blood: $\mu = 0.035\ \frac{g}{cm \cdot s},\ \rho^F = 1\ \frac{g}{cm^3},\ f^F = (0, -g_0 \rho^F),$

$$v_1(x_1, x_2) = \begin{cases} \left(1 - \frac{x_2^2}{H^2}\right) V_0, & (x_1, x_2) \in \Sigma_1 \cup \Sigma_3 \\ V_0, & (x_1, x_2) \in \Sigma_2 \end{cases}$$

$$v_2(x_1, x_2) = 0, \quad (x_1, x_2) \in \Sigma$$

where $V_0 = 30\ \frac{cm}{s}$.

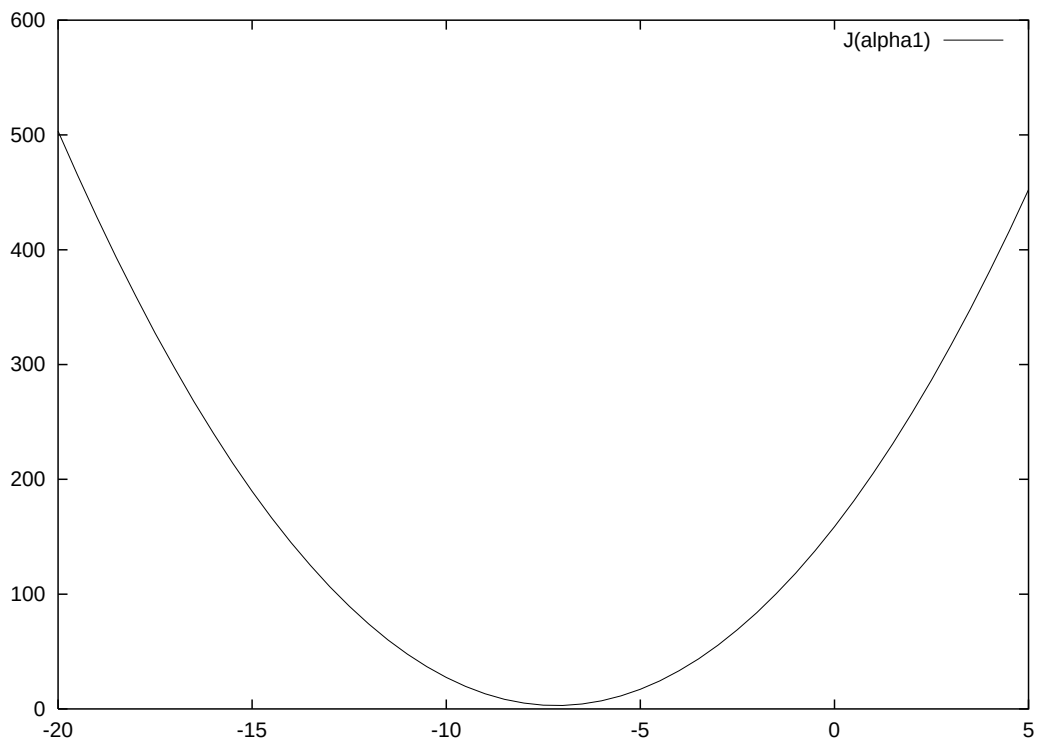
We set $p = 1$ and $\phi_1(x_1) = x_1 - L/2$.

$c_0 = g_0 \rho^S h,\ u_0 = 0,\ c_1 = 0$

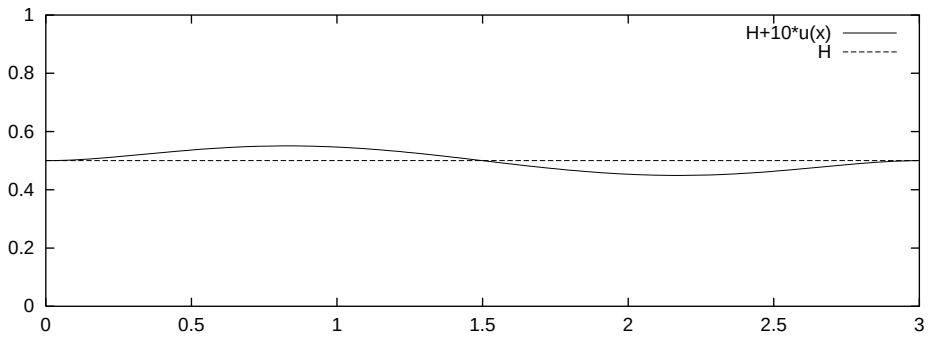
$$u(x_1) = \alpha_1 u_1(x_1),\ u_1(x_1) = \frac{x_1^2 (L - x_1)^2 (x_1 - L/2)}{240 D}.$$

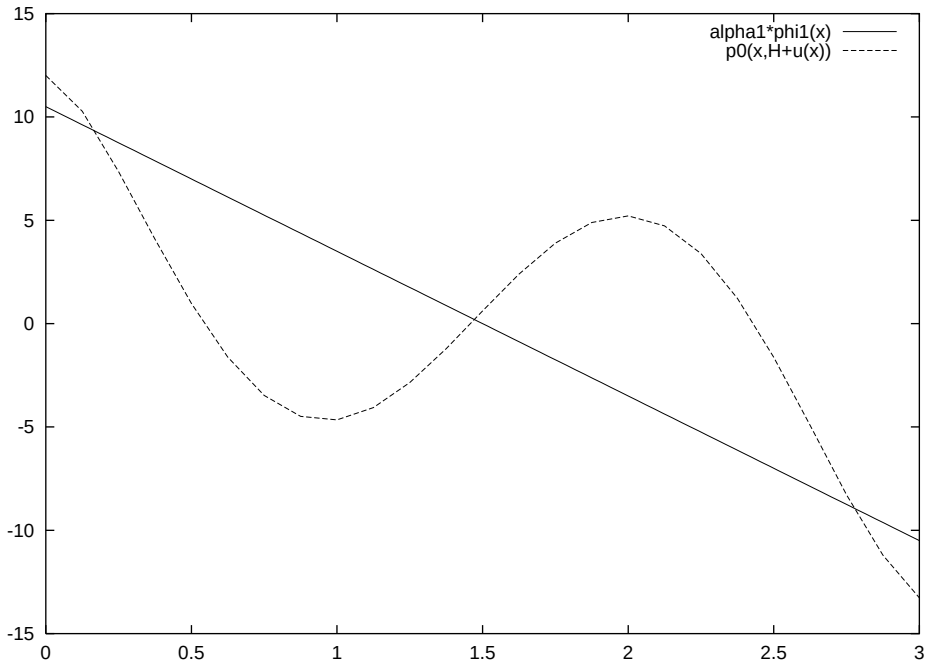
The numerical tests have been produced using *freefem++*.

For the fluid we have used the Mixed Finite Element Method, $P2$ Lagrange triangles for the velocity and $P1$ for the pressure.



The minimal value of the cost function $J = 3.04$ was obtained for $\alpha_1 = -7$.


H+10*u(x)
H
1
0.8
0.6
0.4
0.2
0
0
0.5
1
1.5
2
2.5
3


alpha1*phi1(x)
p0(x,H+u(x))
15
10
5
0
-5
-10
-15
0
0.5
1
1.5
2
2.5
3