Numerical approximation of a pulsatile flow through an flexible pipe

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Content

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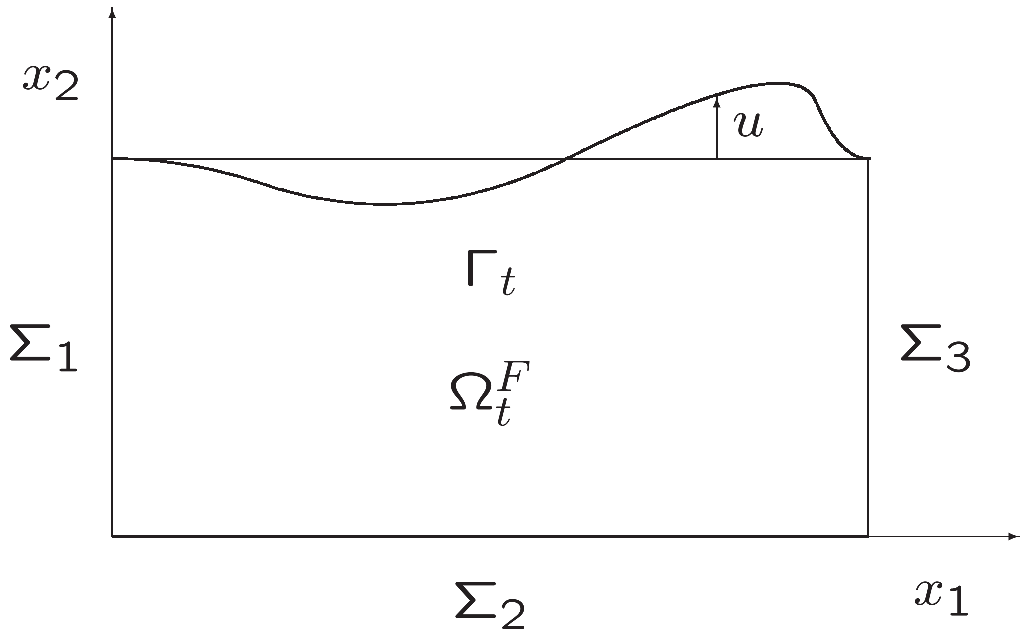
Approximations of the unsteady Navier-Stokes equations in a moving domain. ALE and time discretization. Mixed Finite Element

Approximation of the coupled fluid-structure equations. Identification of the stresses on the interface using the Least Squares Method. Coupled fluid-structure algorithm by BFGS Method

Numerical results

Example of an admissible domain



Strong equations of the structure

Find the transverse displacement $u : [0, L] \times [0, T] \to \mathbb{R}$ such that

$$
\begin{aligned}
&\rho^S h^S \frac{\partial^2 u}{\partial t^2}(x_1, t) + \frac{E(h^S)^3}{12(1-\nu^2)} \frac{\partial^4 u}{\partial x_1^4}(x_1, t) = \eta(x_1, t), \\
&u(0, t) = 0,\ \frac{\partial u}{\partial x_1}(0, t) = 0, \quad t \in (0, T) \\
&u(L, t) = 0,\ \frac{\partial u}{\partial x_1}(L, t) = 0, \quad t \in (0, T) \\
&u(x_1, 0) = u^0(x_1), \quad x_1 \in (0, L) \\
&\frac{\partial u}{\partial t}(x_1, 0) = \dot{u}^0(x_1), \quad x_1 \in (0, L)
\end{aligned}
$$

Strong form of the unsteady Navier-Stokes equations

$$
\begin{aligned}
\rho^F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} \right) - \mu\Delta\mathbf{v} + \nabla p &= \mathbf{f}^F, \quad \forall t \in (0,T), \forall \mathbf{x} \in \Omega_t^F \\
\nabla \cdot \mathbf{v} &= 0, \quad \forall t \in (0,T), \forall \mathbf{x} \in \Omega_t^F \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_1 \times (0,T) \\
p &= P_{in}, \quad \text{on } \Sigma_1 \times (0,T) \\
\mathbf{v} &= \mathbf{g}, \quad \text{on } \Sigma_2 \times (0,T) \\
\mathbf{v} \times \mathbf{n} &= 0, \quad \text{on } \Sigma_3 \times (0,T) \\
p &= P_{out}, \quad \text{on } \Sigma_3 \times (0,T) \\
\mathbf{v}\left(x_1, H + u(x_1,t), t\right) &= \left(0, \frac{\partial u}{\partial t}(x_1,t)\right)^T, \\
&\quad \forall (x_1,t) \in (0,L) \times (0,T) \\
\mathbf{v}(\mathbf{x},0) &= \mathbf{v}^0(\mathbf{x}), \quad \forall \mathbf{x} \in \Omega_0^F
\end{aligned}
$$

Strong form of the coupled equations

Find the transverse displacement u of the structure, the velocity $\mathbf{v}$ and the pressure p of the fluid such that

$$\eta(x_1, t) = -\left(\sigma^F \mathbf{n} \cdot \mathbf{e}_2\right)_{(x_1, H+u(x_1,t))} \sqrt{1 + \left(\frac{\partial u}{\partial x_1}(x_1, t)\right)^2}$$

where $\sigma^F = -p\,I + \mu\left(\nabla \mathbf{v} + \nabla \mathbf{v}^T\right)$ is the stress tensor of the fluid, $\mathbf{e}_2 = (0,1)^T$ is the unit vector in the x_2 direction.

The displacement of the structure depends on the vertical component of the stresses exerced by the fluid on the interface. The movement of the structure changes the domain where the fluid equations must be solved. Also, on the interface we have to impose the equality between the fluid and structure velocity.

Natural frequencies and normal mode shapes

We denote by $0 < s_0 < \cdots < s_i < \ldots$ the solutions of the equation

$$\cos(s)\cosh(s) = 1, \quad s > 0.$$

We set $a_i = s_i/L$. For each $i \in \mathbb{N}$ there exists an unique normal mode shape $\phi_i \in \mathcal{C}^4([0, L])$ such that

$$\begin{aligned}
&\phi_i''''(x_1) = (a_i)^4 \phi_i(x_1), \quad x_1 \in (0, L)\\
&\phi_i(0) = \frac{\partial \phi_i}{\partial x_1}(0) = 0,\\
&\phi_i(L) = \frac{\partial \phi_i}{\partial x_1}(L) = 0,\\
&\int_0^L \phi_i^2(x_1)\, dx_1 = 1.
\end{aligned}$$

Let $\omega_i = \left(\frac{s_i}{L}\right)^2 \sqrt{\frac{E(h^S)^3}{12(1-\nu^2)\rho^S h^S}}$ be the ith natural frequency.

The normal mode shapes ϕ_i for $i \in \mathbb{N}$ form an orthogonal basis of $L^2(0, L)$.

$$\eta(x_1, t) = \sum_{i \geq 0} \alpha_i(t)\phi_i(x_1), \quad u(x_1, t) = \sum_{i \geq 0} q_i(t)\phi_i(x_1)$$

where q_i is the solution of the second order differential equation

$$\begin{aligned}
q_i''(t) + \omega_i^2 q_i(t) &= \frac{1}{\rho^S h^S}\alpha_i(t), \quad t \in (0, T) \\
q_i(0) &= \int_0^L u^0(x_1)\phi_i(x_1)\, dx_1 \\
q_i'(0) &= \int_0^L \dot{u}^0(x_1)\phi_i(x_1)\, dx_1.
\end{aligned}$$

Newmark method

Knowing q_i^n, $\dot{q}_i^n$, $\ddot{q}_i^n$ and α_i^{n+1}, find q_i^{n+1}, $\dot{q}_i^{n+1}$, $\ddot{q}_i^{n+1}$ such that:

$$\ddot{q}_i^{n+1} + \omega_i^2 q_i^{n+1} = \frac{1}{\rho^S h^S}\alpha_i^{n+1},$$
$$\dot{q}_i^{n+1} = \dot{q}_i^n + \Delta t\left[(1-\delta)\ddot{q}_i^n + \delta\ddot{q}_i^{n+1}\right],$$
$$q_i^{n+1} = q_i^n + \Delta t\dot{q}_i^n + (\Delta t)^2\left[\left(\frac{1}{2}-\theta\right)\ddot{q}_i^n + \theta\ddot{q}_i^{n+1}\right]$$

where δ and θ are two real parameters.

$$\left(1+\omega_i^2(\Delta t)^2\theta\right)\ddot{q}_i^{n+1} = \frac{1}{\rho^S h^S}\alpha_i^{n+1} - \omega_i^2\left[q_i^n + \Delta t\dot{q}_i^n + (\Delta t)^2\left(\frac{1}{2}-\theta\right)\ddot{q}_i^n\right].$$

This method is unconditional stable for $2\theta \geq \delta \geq 1/2$. It is first order accuracy if $\delta \neq 1/2$. If $\delta = 1/2$, it is second order accuracy in the case $\theta \neq 1/12$ and forth order accuracy is achieved if $\theta = 1/12$.

Arbitrary Lagrangian Eulerian (ALE) coordinates

$$\widehat{\Omega}^F = (0, L) \times (0, H), \quad \widehat{\Gamma} = (0, L) \times \{H\}$$

$$\mathcal{A}_t(\hat{x}_1, \hat{x}_2) = \left(\hat{x}_1, \frac{H + u(\hat{x}_1, t)}{H}\hat{x}_2\right)^T$$

$$\boldsymbol{\vartheta}^{n+1}(\mathbf{x}) \stackrel{def}{=} \frac{\partial \mathcal{A}_t}{\partial t}(\hat{\mathbf{x}})|_{t=t_{n+1}} = \left(0, \frac{\partial u}{\partial t}\left(x_1, t_{n+1}\right) \frac{x_2}{H + u\left(x_1, t_{n+1}\right)}\right)^T.$$

$$\mathbf{V}^n(\mathbf{x}) = \mathbf{v}^n\left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x})\right)$$

$$
\begin{aligned}
\rho^F \left(\frac{\mathbf{v}^{n+1}}{\Delta t} + \Big((\mathbf{V}^n - \boldsymbol{\vartheta}^{n+1}) \cdot \nabla \Big) \mathbf{v}^{n+1} \right) & \\
-\mu \Delta \mathbf{v}^{n+1} + \nabla p^{n+1} &= \rho^F \frac{\mathbf{V}^n}{\Delta t} + \mathbf{f}^F \\
& \quad \text{in } \Omega^F_{t_{n+1}} \\
\nabla \cdot \mathbf{v}^{n+1} &= 0 \text{ in } \Omega^F_{t_{n+1}} \\
\mathbf{v}^{n+1} \times \mathbf{n} &= 0 \text{ on } \Sigma_1 \\
p^{n+1} &= P_{in}(\cdot, t_{n+1}) \text{ on } \Sigma_1 \\
\mathbf{v}^{n+1} &= \mathbf{g}(\cdot, t_{n+1}) \text{ on } \Sigma_2 \\
\mathbf{v}^{n+1} \times \mathbf{n} &= 0 \text{ on } \Sigma_3 \\
p^{n+1} &= P_{out}(\cdot, t_{n+1}) \text{ on } \Sigma_3 \\
\mathbf{v}^{n+1} \Big(x_1, H + u(x_1, t_{n+1}), t \Big) &= \left(0, \frac{\partial u}{\partial t}(x_1, t_{n+1}) \right)^T, \\
& \quad 0 < x_1 < L.
\end{aligned}
$$

The optimization approach

If the starting point is not chosen "sufficiently close" to the solution, fixed point or Newton like methods diverge.

The continuity of the stresses on the interface will be treated by the Least Square Method and at each time step we have to solve an optimization problem which is less sensitive to the choice of the starting point. This is the main advantage of this approach.

Another attractive point is that the fluid and structure meshes do not necessarily match on the interface.

Our approach is to minimize

$$J^{n+1}(\boldsymbol{\alpha}) = \frac{1}{2}\|\boldsymbol{\alpha} - \boldsymbol{\beta}\|_2^2 = \frac{1}{2}\|\boldsymbol{\alpha} - \mathcal{F} \circ \mathcal{S}(\boldsymbol{\alpha})\|_2^2$$

The fixed point framework is to solve

$$\mathcal{F} \circ \mathcal{S}(\boldsymbol{\alpha}) = \boldsymbol{\alpha}.$$

Identification of the stresses on the interface using the Least Squares Method

The stresses on the interface at the current time step t_{n+1} will be approached by $\eta_m^{n+1}(x_1) = \sum_{i=0}^{m-1} \alpha_i^{n+1}\phi_i(x_1)$.

The parameters α_i^{n+1} for $0 \leq i \leq m-1$ will be "identified" solving an optimization problem, more precisely

$$\boldsymbol{\alpha}^{n+1} \stackrel{def}{=} \left(\alpha_0^{n+1}, \ldots, \alpha_{m-1}^{n+1}\right) \in \arg\min_{\boldsymbol{\alpha}\in\mathbb{R}^m} J^{n+1}(\boldsymbol{\alpha}).$$

Let $\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{m-1}) \in \mathbb{R}^m$.

Structure sub-problem

For $i = 0, \ldots, m-1$, knowing q_i^n, $\dot{q}_i^n$, $\ddot{q}_i^n$, find Q_i, $\dot{Q}_i$, $\ddot{Q}_i$ such that:

$$\left(1 + \omega_i^2 (\Delta t)^2 \theta\right) \ddot{Q}_i = \frac{1}{\rho^S h^S} \alpha_i - \omega_i^2 \left[q_i^n + \Delta t \dot{q}_i^n + (\Delta t)^2 \left(\frac{1}{2} - \theta \right) \ddot{q}_i^n \right]$$

$$\dot{Q}_i = \dot{q}_i^n + \Delta t \left[(1-\delta) \ddot{q}_i^n + \delta \ddot{Q}_i \right],$$

$$Q_i = q_i^n + \Delta t \dot{q}_i^n + (\Delta t)^2 \left[\left(\frac{1}{2} - \theta \right) \ddot{q}_i^n + \theta \ddot{Q}_i \right]$$

Set

$$U(x_1) = \sum_{i=0}^{m-1} Q_i \phi_i(x_1), \; \dot{U}(x_1) = \sum_{i=0}^{m-1} \dot{Q}_i \phi_i(x_1), \; \ddot{U}(x_1) = \sum_{i=0}^{m-1} \ddot{Q}_i \phi_i(x_1).$$

Fluid sub-problem

$$\begin{aligned} W_h &= \Big\{ \mathbf{w}_h \in \left(C^0\left(\overline{\Omega}_h^F\right)\right)^2;\ \forall K \text{ triangle of } \mathcal{T}_h,\ \mathbf{w}_h|_K \in P1 + bubble, \\ &\qquad \mathbf{w}_h \times \mathbf{n} = 0 \text{ on } \Sigma_1 \cup \Sigma_3,\ \mathbf{w}_h = 0 \text{ on } \Sigma_2 \cup \Gamma_h \Big\}, \\ Q_h &= \Big\{ q_h \in C^0\left(\overline{\Omega}_h^F\right);\ \forall K \text{ triangle of } \mathcal{T}_h,\ q_h|_K \in P1 \Big\}. \end{aligned}$$

Find the velocity $\mathbf{v}_h$ satisfies the boundary conditions

$$\begin{aligned} \mathbf{v}_h \times \mathbf{n} &= 0,\ on\ \Sigma_1 \cup \Sigma_3, \\ \mathbf{v}_h &\approx \mathbf{g}(\cdot, t_{n+1})\ on\ \Sigma_2, \\ \mathbf{v}_h\left(x_1, H + U(x_1)\right) &\approx \left(0, \dot{U}(x_1)\right)^T,\ 0 < x_1 < L \end{aligned}$$

and the pressure $p_h \in Q_h$ such that

$$\begin{cases} a_F^{n+1}\left(\mathbf{v}_h, \mathbf{w}_h\right) + d_F^{n+1}\left(\mathbf{v}_h, \mathbf{w}_h\right) + b_F^{n+1}\left(\mathbf{w}_h, p_h\right) = \ell^{n+1}\left(\mathbf{w}_h\right), & \forall \mathbf{w}_h \in W_h \\ b_F^{n+1}\left(\mathbf{v}_h, q_h\right) = 0, & \forall q_h \in Q_h \end{cases}$$

Definition of the cost function

Let us define for $i = 0, \ldots, m-1$

$$\beta_i = -\int_0^L \phi_i(x_1) \left(\sigma^F (\mathbf{v}_h, p_h) \, \mathbf{n} \cdot \mathbf{e}_2 \right)_{(x_1, H+U(x_1))} \sqrt{1 + \left(\frac{\partial U}{\partial x_1}(x_1) \right)^2} dx_1.$$

$$\beta_i = \int_0^L \phi_i(x_1) \left(p_h - \mu \left(\frac{\partial v_{h,1}}{\partial x_2} + \frac{\partial v_{h,2}}{\partial x_1} \right) \left(-\frac{\partial U}{\partial x_1} \right) - 2\mu \frac{\partial v_{h,2}}{\partial x_2} \right)_{(x_1, H+U(x_1))} dx_1.$$

Set the cost function

$$J^{n+1}(\boldsymbol{\alpha}) = \frac{1}{2} \sum_{i=0}^{m-1} (\alpha_i - \beta_i)^2 .$$

$$\beta_i = \int_0^L \phi_i(x_1) p_h \left(x_1, H + U(x_1) \right) dx_1, \quad i = 0, \ldots, m-1.$$

Numerical results

The computation has been made in a domain of length $L = 6\ cm$ and height $H = 1\ cm$.

The viscosity of the blood was taken to be $\mu = 0.035\ \frac{g}{cm \cdot s}$, its density $\rho^F = 1\ \frac{g}{cm^3}$. The thickness of the vessel is $h = 0.1\ cm$, the Young modulus $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the Poisson ratio $\nu = 0.5$, the density $\rho^S = 1.1\ \frac{g}{cm^3}$.

The numerical tests have been produced using *freefem++ v1.34*.

We have performed the simulation for $N = 500$ time steps with a time step $\Delta t = 0.0005\ s$ which gives a time duration $T = N\Delta t = 0.25\ s$.

Case of a sine wave of the pressure input in a less compliant vessel

The Young modulus: $E = 3 \cdot 10^6 \ \frac{g}{cm \cdot s^2}$. The pressure at the inflow:

$$P_{in}(\mathbf{x}, t) = \begin{cases} 10^3(1 - \cos(2\pi t/0.025)), & \mathbf{x} \in \Sigma_1, 0 \le t \le 0.025 \\ 0, & \mathbf{x} \in \Sigma_1, 0.005 \le t \le T \end{cases}$$

	Δt	h	N	T	$\max_n J^{n+1}(\boldsymbol{\alpha}^n)$	$\max_n J^{n+1}(\boldsymbol{\alpha}^{n+1})$
Test A	0.0005	0.25	200	0.1	$1.8 \cdot 10^5$	0.00060
Test B	0.0010	0.17	100	0.1	$5 \cdot 10^5$	0.00055
Test C	0.0025	0.10	40	0.1	$1.4 \cdot 10^6$	0.00055

We have performed 10 iterations of the BFGS algorithm and 5 iterations in the method for the line search.

Conclusions

The continuity of the stresses at the interface was treated by the Least Squares Method and at each time step we have to solve an optimization problem which is less sensitive to the choice of the starting point and it permits us to use moderate time step. This is the main advantage of this approach.

In order to solve the optimization problem, we have employed the BFGS method which is successful from farther starting point. The gradient of the cost function was approached by the Finite Difference Method.

The coupled fluid-structure algorithm has good stability properties.