Mixed hybrid finite element for a three dimensional fluid structure interaction problem

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Abstract

In this paper, we study the spatial discretization of a fluid structure interaction problem by using mixed hybrid finite elements. The aim is to prove that the fully discretized problem is well posed for suitable finite elements. In order to show that the *inf-sup* discrete condition holds, we construct a projection operator.

keywords. mixed hybrid finite element; fluid structure interaction

1 INTRODUCTION

A three dimensional fluid structure interaction problem is studied under the following hypotheses: the fluid is incompressible and limited by an elastic structure, the whole interior cavity of the structure is filled by the fluid, the structure is thick. A part of the external boundary is fixed.

This kind of fluid structure interaction concerns many important domains: biomechanics (blood cardiac wall interaction), aeronautical industry (the interaction between the tank and the fuel of the airship), energy industry (transport of the fluid using elastic tanks).

We are interested in computing the displacement of the structure, the velocity and the pressure of the fluid and the density of the forces on the interface.

We suppose that the structure is governed by the time dependent linear elasticity equations and the fluid is governed by the time dependent Stokes equations.

Based on the works [1] and [2, vol. 8, p. 795] and under the hypotheses above, a variational formulation is proposed in [3]. The existence and the uniqueness of the solution of this variational problem are proved in [4]. In this new formulation a Lagrange multiplier was used to relax the continuity of the velocities on the interface and it permits to split the fluid equations from the structure equations. The advantage lies in the fact that we can solve the

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fluid structure interaction via partitioned procedures. Also, this model is well adapted for parallel computation.

In order to approximate the solution, we had first discretized in time using Finite Difference Method. The time discretization corresponds to the implicit Euler method for the fluid equations and Newmark method for the structure equations. In [5] it is proved that the time discrete problem is well posed. Also the stability in time of the semi-discrete algorithm is proved.

At each time step, we have to solve a mixed hybrid system with two Lagrange multipliers which treat the free divergence for the fluid and the continuity of the velocities on the interface.

In this paper we study the spatial discretization of this mixed hybrid variational problem.

The aim is to present the choice for the three dimensional mixed-hybrid finite element and to prove that the fully discretized problem is well posed. It is not our purpose to study the stability of the spatial discretization in this paper.

In order to approximate each component of the fluid velocity, we use the MIDI element (piecewise linear with bubble) introduced in [6]. The discrete solution for the pressure is piecewise linear. Also, each component of the structure velocity and each component of the forces on the interface are approximated by piecewise linear elements. All the discrete solutions are assumed to be globally continuous.

In order to prove that the *inf-sup* discrete condition holds, we construct a projection operator. The *inf-sup* discrete condition involves the free divergence for the fluid and the continuity of the velocities on the interface. The *inf-sup* discrete condition for the free divergence is a standard result (see for example [7]). The proof for the *inf-sup* discrete condition for coupling equations was already discussed in the papers [8], [9], [10] and [11], but unlike ours, it doesn't involve fluid structure interaction.

2 GOVERNING EQUATIONS

Let Ω^F (respectively Ω^S) be the domain in $\mathbb{R}^N$ of the fluid (respectively of the structure), where $N = 3$, such that $\partial\Omega^F = \overline{\Gamma} \cup \overline{\Gamma^1}$, $\partial\Omega^S = \overline{\Gamma} \cup \overline{\Sigma^1} \cup \overline{\Sigma^2}$ and $\overline{\Omega^F} \cap \overline{\Omega^S} = \overline{\Gamma}$, where Γ, Γ^1, Σ^1 and Σ^2 are manifolds in $\mathbb{R}^{N-1}$. The geometrical configuration is presented in the Figure 1.

We suppose that the fluid is governed by the time-dependent Stokes equations:

$$\rho_F \frac{\partial v}{\partial t} - \mu_F \Delta v + \nabla p = f^1 \qquad \text{in } \Omega^F \times]0,T[\tag{1}$$

$$\text{div } v = 0 \qquad \text{in } \Omega^F \times]0,T[\tag{2}$$

where $v : \overline{\Omega^F} \times [0,T] \to \mathbb{R}^N$ is the velocity vector, $p : \overline{\Omega^F} \times [0,T] \to \mathbb{R}$ is the pressure, $\rho_F > 0$ is the density, $\mu_F > 0$ is the viscosity and $f^1 : \overline{\Omega^F} \times [0,T] \to \mathbb{R}^N$ is the body force per unit mass.

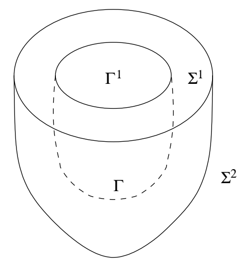

Figure 1: The interface between the fluid and the structure is Γ.

The structure satisfy the linear elasticity equations:

$$\rho_S \frac{\partial^2 u_i}{\partial t^2} - \sum_{j=1}^{N} \frac{\partial \sigma_{ij}^S(u)}{\partial x_j} = f_i^2, \quad \text{in } \Omega^S \times]0,T[, \quad i = 1, \ldots, N \tag{3}$$

where $u = (u_1, u_2, u_3) : \overline{\Omega^S} \times [0,T] \to \mathbb{R}^N$ is the displacement of the structure, $\rho_S > 0$ is the density, $\sigma_{ij}^S(u) = \lambda_S \big(\sum_{k=1}^N \epsilon_{kk}(u)\big)\delta_{ij} + 2\mu_S \epsilon_{ij}(u)$ is the structure stress tensor, $\epsilon_{ij}(u) = \frac{1}{2}\big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\big)$ is the deformation tensor, $\lambda_S \geq 0$, $\mu_S > 0$ are the Lamé's coefficients and $f^2 = (f_1^2, f_2^2, f_3^2) : \overline{\Omega^S} \times [0,T] \to \mathbb{R}^N$ is the body force per unit mass.

We denote by n^1 (respectively n^2) the unit outward normal on the boundary of Ω^F (respectively Ω^S). We impose the following boundary conditions:

$$v = g \qquad \text{on } \Gamma^1 \times]0,T[, \tag{4}$$
$$u = 0 \qquad \text{on } \Sigma^1 \times]0,T[, \tag{5}$$
$$\sigma^S \cdot n^2 = 0 \qquad \text{on } \Sigma^2 \times]0,T[, \tag{6}$$

where g is given such that $\int_{\Gamma^1} g \cdot n^1 \, d\sigma = 0$ and on the interface, we have:

$$v = \frac{\partial u}{\partial t} \qquad \text{on } \Gamma \times]0,T[, \tag{7}$$
$$\sigma^F \cdot n^1 = -\sigma^S \cdot n^2 \qquad \text{on } \Gamma \times]0,T[, \tag{8}$$

where $\sigma^F = -pI + 2\mu_F D[v]$ is the fluid stress tensor and $D[v]_{ij} = \frac{1}{2}\big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\big)$.

The initial conditions are:

$$v(\cdot, t=0) = v^0 \qquad \text{in } \Omega^F, \tag{9}$$
$$u(\cdot, t=0) = u^0 \qquad \text{in } \Omega^S, \tag{10}$$
$$\frac{\partial u}{\partial t}(\cdot, t=0) = \nu^0 \qquad \text{in } \Omega^S, \tag{11}$$

where v^0 should be divergence-free, $v^0 = g$ on Γ^1 and $v^0 = \nu^0$ on Γ.

We are looking for the displacement of the structure u, the velocity and the pressure of the fluid v, p, such that (1)–(11) hold.

3 TEMPORAL DISCRETIZATION

Let Δt be the time step size. For the fluid equations (1)–(2), we use the backward Euler time discretization

$$\begin{aligned} \rho_F \frac{v^{n+1} - v^n}{\Delta t} - \mu_F \Delta v^{n+1} + \nabla p^{n+1} &= f^1\left(\cdot, (n+1)\Delta t\right) && (12) \\ \text{div } v^{n+1} &= 0 && (13) \end{aligned}$$

where v^n and p^n are the approximations of v and p at time $t = n\Delta t$.

For the structure equation (3), we use the Newmark scheme

$$\begin{aligned} \rho_S\, a_i^{n+1} - \sum_{j=1}^{N} \frac{\partial \sigma_{ij}^S(u^{n+1})}{\partial x_j} &= f_i^2\left(\cdot, (n+1)\Delta t\right), \quad i = 1, 2, 3 && (14) \\ u^{n+1} &= u^n + \frac{\Delta t}{2}\left(\nu^n + \nu^{n+1}\right) && (15) \\ a^{n+1} &= \frac{\nu^{n+1} - \nu^n}{\Delta t} && (16) \end{aligned}$$

where u^n, ν^n and a^n are the approximations of u, $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial t^2}$ at time $t = n\Delta t$.

Repeated application of the formula (15) enables us to write

$$u^{n+1} = u^0 + \frac{\Delta t}{2}\left(\nu^0 + 2\sum_{i=1}^{n} \nu^n + \nu^{n+1}\right).$$

Finally, substituting (16) and the above formula in (14), we obtain the scheme

$$\rho_S \frac{\nu_i^{n+1} - \nu_i^n}{\Delta t} - \sum_{j=1}^{N} \frac{\partial \sigma_{ij}^S\left(u^0 + \frac{\Delta t}{2}\left(\nu^0 + 2\sum_{i=1}^{n} \nu^n + \nu^{n+1}\right)\right)}{\partial x_j} = f_i^2\left(\cdot, (n+1)\Delta t\right) \tag{17}$$

Now, we present the weak formulation of the semidiscretized problem.

In [3], a Lagrange multiplier λ was used to relax the condition (7) which represents the continuity of the velocities on the interface. This Lagrange multiplier has the physical signification of the density of the forces on the interface ($\lambda = \sigma^F \cdot n^1 = -\sigma^S \cdot n^2$) and it permits to split the fluid equations from the structure equations.

We denote respectively by

$$\begin{aligned} W^1 &= \left\{w^1 \in H^1(\Omega^F)^N,\ w^1 = 0 \text{ on } \Gamma^1\right\}, \\ W^2 &= \left\{w^2 \in H^1(\Omega^S)^N,\ w^2 = 0 \text{ on } \Sigma^1\right\}, \\ Q &= L^2(\Omega^F), \\ M &= H_{00}^{1/2}(\Gamma)^N, \end{aligned}$$

the spaces for the velocity of the fluid, for the velocity of the structure, for the pressure of the fluid and for the forces on the interface.

Let a_F and a_S be two bilinear applications, defined by:

$$
\begin{aligned}
a_F : W^1 \times W^1 \longrightarrow \mathbb{R}, \\
a_F(v, w^1) = 2\int_{\Omega^F} D[v] : D[w^1] dx = 2\sum_{i,j=1}^{N} \int_{\Omega^F} D[v]_{ij} D[w^1]_{ij} dx
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
a_S : W^2 \times W^2 \longrightarrow \mathbb{R}, \\
a_S(\nu, w^2) = \sum_{i,j=1}^{N} \int_{\Omega^S} \sigma_{ij}^S(\nu)\epsilon_{ij}(w^2) dx.
\end{aligned}
\tag{19}
$$

Remark 1 *In [1] and [2, vol. 8, p. 795], it was used the bilinear form*

$$
a_F(v, w^1) = \int_{\Omega^F} \nabla v \cdot \nabla w^1 dx.
$$

In this case, the weak solution of the fluid beam interaction doesn't satisfy (8) on the interface, but

$$
-pn^1 + \mu_F \frac{\partial v}{\partial n^1} = -\sigma^S \cdot n^2, \quad \text{on } \Gamma \times]0, T[.
$$

Let us denote by a and b the functions given by

$$
\begin{aligned}
a : \left(W^1 \times W^2\right) \times \left(W^1 \times W^2\right) \longrightarrow \mathbb{R}, \\
a\left((v, \nu); (w^1, w^2)\right) = \frac{1}{\Delta t}(v, w^1)_{0,\Omega^F} + \frac{\mu_F}{\rho_F} a_F(v, w^1) \\
+\frac{1}{\Delta t}(\nu, w^2)_{0,\Omega^S} + \frac{\Delta t}{2\rho_S} a_S(\nu, w^2),
\end{aligned}
\tag{20}
$$

and

$$
\begin{aligned}
b : \left(W^1 \times W^2\right) \times (Q \times M) \longrightarrow \mathbb{R}, \\
b\left(\left(w^1, w^2\right); (q, \mu)\right) = -(\text{div } w^1, q)_{0,\Omega^F} - \left(\gamma_\Gamma^1(w^1) - \gamma_\Gamma^2(w^2), \mu\right)_{1/2,\Gamma},
\end{aligned}
\tag{21}
$$

where

$$
\gamma_\Gamma^1 : W^1 \longrightarrow M, \qquad \gamma_\Gamma^2 : W^2 \longrightarrow M
$$

are the trace applications.

In [5] it was introduced a time discrete algorithm for the approximation of a three-dimensional fluid-structure interaction. This algorithm consists in solving at each time step a problem of the following form:

Find $\left(v^{n+1}, \nu^{n+1}, p^{n+1}, \lambda^{n+1}\right) \in W^1 \times W^2 \times Q \times M$, *such that:*

$$
\left\{
\begin{array}{lcl}
a\left(v^{n+1}, \nu^{n+1}; w^1, w^2\right) + b\left(w^1, w^2; p^{n+1}, \lambda^{n+1}\right) & = & \langle f^{n+1}; w^1, w^2 \rangle \\
b\left(v^{n+1}, \nu^{n+1}; q, \mu\right) & = & 0,
\end{array}
\right.
\tag{22}
$$

for all w^1 in W^1, w^2 in W^2, q in Q and μ in M, where

$$\begin{aligned}\langle f^{n+1}; w^1, w^2\rangle &= (f_1^{n+1}, w^1)_{0,\Omega^F} + (f_2^{n+1}, w^2)_{0,\Omega^S} + \frac{1}{\Delta t}(v^n, w^1)_{0,\Omega^F} + \frac{1}{\Delta t}(\nu^n, w^2)_{0,\Omega^S} \\ &\quad - \frac{1}{\rho_S} a_S\Big(u_0 + \frac{\Delta t}{2}(\nu^0 + 2\sum_{i=1}^{n}\nu^i), w^2\Big)\end{aligned}$$

In [5] it is shown that the time discrete problem is well posed and it is also proved the stability in time of the semi-discrete algorithm.

Remark 2 *At each time step, we have to solve the system with Lagrange multipliers (22). This is a Babuska-Brezzi type variational problem (see [12] and [13]), where the Lagrange multipliers are p and λ, which treat respectively the free-divergence for the fluid and the continuity of the velocity on the interface.*

4 THE CHOICE OF THE MIXED-HYBRID FINITE ELEMENT

We construct four finite-dimensional spaces using mixed-hybrid finite elements, such that the *inf-sup* discrete condition holds.

Let us suppose that Ω^F and Ω^S are two bounded polyhedrons of $\mathbb{R}^3$.

We consider the finite element like triangulations

$$\overline{\Omega^F} = \bigcup_{K\in\mathcal{T}_h^F} \overline{K} \text{ and } \overline{\Omega^S} = \bigcup_{K\in\mathcal{T}_h^S} \overline{K}$$

where each element K is an open nondegenerate tetrahedron and meshes $\mathcal{T}_h^F$ and $\mathcal{T}_h^S$ are matching on the interface $\overline{\Gamma} = \overline{\Omega^F} \cap \overline{\Omega^S}$. The diameter of each element is less than $h > 0$.

Before presenting the finite element chosen, let us recall the barycentric coordinates of a point from $\mathbb{R}^3$ with respect to the vertices of a tetrahedron.

Let K be a nondegenerate tetrahedron of vertices $a_i \in \mathbb{R}^3$, $1 \le i \le 4$.

We denote by P_1 (respectively $P_1(K)$) the space of all polynomials defined on $\mathbb{R}^3$ (respectively on K) of degree less than or equal to one.

The barycentric coordinates $\lambda_i : \mathbb{R}^3 \longrightarrow \mathbb{R}$, $1 \le i \le 4$, with respect to the vertices $a_i \in \mathbb{R}^3$, $1 \le i \le 4$, are defined by

$$\begin{gathered}\lambda_i \in P_1, \quad 1 \le i \le 4, \\ \lambda_i(a_j) = \delta_{ij}, \quad 1 \le i, j \le 4.\end{gathered}$$

For the tetrahedron K, one can define (see [7, p. 147]) the bubble function

$$\begin{gathered}b_K : K \longrightarrow \mathbb{R}, \\ b_K(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x)\lambda_4(x).\end{gathered}$$

We denote

$$P_1(K) + b_K = \{q : K \to \mathbb{R},\ q(x) = p_1(x) + \alpha b_K(x), \quad p_1 \in P_1(K),\ \alpha \in \mathbb{R}\}.$$

Let a_5 denote the barycentre of K, i.e.

$$a_5 = \frac{1}{4}(a_1 + a_2 + a_3 + a_4).$$

Classical results assert that the following 3-uples

$$\begin{gathered}(K, \{a_1, a_2, a_3, a_4\}, P_1(K)), \\ (K, \{a_1, a_2, a_3, a_4, a_5\}, P_1(K) + b_K)\end{gathered}$$

are finite elements.

If T is a non degenerate triangle in $\mathbb{R}^3$, we denote by $P_1(T)$ the space of all polynomials defined on $\mathbb{R}^3$ of degree less than or equal to one.

Now, we can introduce the finite element spaces:

$$\begin{aligned}
W_h^1 &= \left\{w_h^1 = \left(w_h^{1,1}, w_h^{1,2}, w_h^{1,3}\right);\ w_h^{1,i} \in \mathcal{C}^0\left(\overline{\Omega^F}\right), w_h^{1,i}\mid_K \in P_1(K) + b_K, 1 \le i \le 3\right\}, \\
W_h^2 &= \left\{w_h^2 = \left(w_h^{2,1}, w_h^{2,2}, w_h^{2,3}\right);\ w_h^{2,i} \in \mathcal{C}^0\left(\overline{\Omega^S}\right), w_h^{2,i}\mid_K \in P_1(K), w_h^{2,i}\mid_{\Sigma^1} = 0, 1 \le i \le 3\right\}, \\
Q_h &= \left\{q_h \in \mathcal{C}^0\left(\overline{\Omega^F}\right);\ q_h\mid_K \in P_1(K)\right\}, \\
M_h &= \left\{\mu_h = \left(\mu_h^1, \mu_h^2, \mu_h^3\right);\ \mu_h^i \in \mathcal{C}^0\left(\overline{\Gamma}\right),\ \mu_h^i\mid_T \in P_1(T),\ 1 \le i \le 3\right\}.
\end{aligned}$$

In a standard way (see for example [14, p. 27]), we have the inclusions

$$W_h^1 \subseteq W^1, \quad W_h^2 \subseteq W^2, \quad Q_h \subseteq Q, \quad M_h \subseteq M.$$

The continuity of the elements of M_h seems not to be appropriate, at least when edges or corners appear. But it is necessary for the internal approximation of $H_{00}^{1/2}(\Gamma)^N$, that is, having discontinuous trial functions, we lose the inclusion $M_h \subseteq M$.

One way to check the *inf-sup* discrete condition is by using that the *inf-sup* continuous condition holds and by constructing a bilinear operator Π_h from $W^1 \times W^2$ to $W_h^1 \times W_h^2$, such that

$$\left\{\begin{array}{c}\exists c > 0, \forall \left(w^1, w^2\right) \in W^1 \times W^2, \\ \left\|\Pi_h\left(w^1, w^2\right)\right\|_{W^1 \times W^2} \le c\left\|\left(w^1, w^2\right)\right\|_{W^1 \times W^2}\end{array}\right. \tag{23}$$

and

$$\left\{\begin{array}{c}\forall \left(w^1, w^2\right) \in W^1 \times W^2, \forall \left(q_h, \mu_h\right) \in Q_h \times M_h, \\ b\left(\left(w^1, w^2\right) - \Pi_h\left(w^1, w^2\right); \left(q_h, \mu_h\right)\right) = 0.\end{array}\right. \tag{24}$$

Remark 3 *Let us remark that in general the constant c depends on the mesh size h, which implies that the discrete problem is well posed. If the interpolation operator Π_h is uniformly continuous with respect to h, then the stability of the spatial discretizations holds. It is not our purpose to study the stability in this paper.*

In [5], it is proved that the *inf-sup* continuous condition holds. In order to prove that the *inf-sup* discrete condition holds too, it is sufficient to construct an operator with the above properties.

5 CONSTRUCTION OF THE INTERPOLATION OPERATOR

In this section, we construct an interpolation operator Π_h, such that the equalities (23) and (24) hold.

We search an operator Π_h of the form

$$\begin{gathered}\Pi_h : W^1 \times W^2 \to W_h^1 \times W_h^2, \\ \Pi_h\left(w^1, w^2\right) = \left(\Pi_h^{11} w^1, \Pi_h^{21} w^1 + \Pi_h^{22} w^2\right)\end{gathered}$$

where the operators

$$\begin{gathered}\Pi_h^{11} : W^1 \to W_h^1, \\ \Pi_h^{21} : W^1 \to W_h^2, \quad \Pi_h^{22} : W^2 \to W_h^2\end{gathered}$$

are linear.

The condition (24) could be rewritten as follows

$$\left\{\begin{array}{lcl}\forall w^1 \in W^1, \forall w^2 \in W^2, \forall q_h \in Q_h, \forall \mu_h \in M_h, & & \\ \left(\operatorname{div}\ \left(w^1 - \Pi_h^{11} w^1\right), q_h\right)_{0,\Omega^F} & = & 0, \\ \left(\gamma_\Gamma^1\left(w^1 - \Pi_h^{11} w^1\right) + \gamma_\Gamma^2\left(\Pi_h^{21} w^1\right), \mu_h\right)_{1/2,\Gamma} & = & 0, \\ \left(\gamma_\Gamma^2\left(w^2 - \Pi_h^{22} w^2\right), \mu_h\right)_{1/2,\Gamma} & = & 0.\end{array}\right. \tag{25}$$

Proposition 1 *If the mesh $\mathcal{T}_h^F$ is uniform, there is a linear operator Π_h^{11} from W^1 to W_h^1 and a positive constant c_{11} independent of h, such that*

$$\forall w^1 \in W^1, \left\|\Pi_h^{11} w^1\right\|_{W^1} \leq c_{11}\left\|w^1\right\|_{W^1} \tag{26}$$

and

$$\forall w^1 \in W^1, \forall q_h \in Q_h, \left(div\ \left(w^1 - \Pi_h^{11} w^1\right), q_h\right)_{0,\Omega^F} = 0. \tag{27}$$

Proof. We can find a proof of this standard result in [6] for instance.

Before constructing the operators Π_h^{21} and Π_h^{22}, let us introduce same notations.

We denote by
$I_F = \{A_i\}_{i=1}^{NSF}$ the set of the vertex of the triangulation $\mathcal{T}_h^F$,
$I_S = \{B_i\}_{i=1}^{NSS}$ the set of the vertex of the triangulation $\mathcal{T}_h^S$,
$I_\Gamma = \{C_i\}_{i=1}^{NSG}$ the set of the common vertex of $\mathcal{T}_h^F$ and $\mathcal{T}_h^F$,
I_{Σ^1} the set of the vertex of $\mathcal{T}_h^S \cap \overline{\Sigma^1}$.

We will denote

$$\Theta_h = \left\{\theta_h \in \mathcal{C}^0\left(\overline{\Omega^F}\right); \theta_h\mid_K \in P_1(K), \forall K \in \mathcal{T}_h^F\right\}. \tag{28}$$

It is clear that for each vertex $A_i \in I_F$, there exists a unique function $\phi_i \in \Theta_h$, such that

$$\phi_i\left(A_j\right) = \delta_{ij},\ \forall A_j \in I_F.$$

Moreover, $\{\phi_i\}_{i=1}^{NSF}$ represents a basis for Θ_h (see [14, p. 107]).

We set

$$(P_1 + b)\left(\Omega^F\right) = \left\{ w \in \mathcal{C}^0\left(\overline{\Omega^F}\right); w\mid_K \in P_1(K), \forall K \in \mathcal{T}_h^F \right\}. \tag{29}$$

According to the definition of the bubble function, the set $\{\phi_i\}_{i=1}^{NSF} \cup \{b_K\}_{K\in\mathcal{T}_h^F}$ is a basis for $(P_1 + b)\left(\Omega^F\right)$.

We denote

$$\Psi_h = \left\{ \psi_h \in \mathcal{C}^0\left(\overline{\Omega^F}\right); \psi\mid_{\Sigma^1} = 0, \psi\mid_K \in P_1(K), \forall K \in \mathcal{T}_h^S \right\}. \tag{30}$$

For each vertex $B_i \in I_S \setminus I_{\Sigma^1}$ there exists a unique function $\psi_i \in \Psi_h$, such that

$$\psi_i(B_j) = \delta_{ij},\ \forall B_j \in I_S.$$

It follows that $\{\psi_i\}_{i=1}^{NSS}$ is a basis for Ψ_h.

We define the set

$$\Upsilon_h = \left\{ \eta \in \mathcal{C}^0\left(\overline{\Gamma}\right); \eta\mid_T \in P_1(T), \forall T \in \bigcup_{K\in\mathcal{T}_h^F} \left(K \bigcap \overline{\Gamma}\right) \right\}. \tag{31}$$

For each vertex $C_i \in I_\Gamma$, there exists a unique function $\eta_i \in \Upsilon_h$, such that

$$\eta_i(C_j) = \delta_{ij},\ \forall C_j \in I_\Gamma.$$

Moreover, $\{\eta_i\}_{i=1}^{NSG}$ represents a basis for Υ_h.

Since $\mathcal{T}_h^F$ and $\mathcal{T}_h^S$ are matching on the interface Γ, we have

$$\begin{array}{c} \forall i \in \{1,\ldots,NSG\}, \\ \exists!\ r(i) \in \{1,\ldots,NSF\}, \exists!\ s(i) \in \{1,\ldots,NSS\},\ \text{such that} \\ C_i = A_{r(i)} = B_{s(i)} \end{array} \tag{32}$$

and

$$\eta_i = \phi_{r(i)}\mid_\Gamma = \psi_{s(i)}\mid_\Gamma. \tag{33}$$

Now, we can construct the linear operator Π_h^{21}.

Proposition 2 *There is a linear operator* Π_h^{21} *from* W^1 *to* W_h^2 *and a positive constant* $c_{21}(h)$ *depending of* h*, such that*

$$\forall w^1 \in W^1, \left\|\Pi_h^{21} w^1\right\|_{W^2} \leq c_{21}(h)\left\|w^1\right\|_{W^1} \tag{34}$$

and

$$\begin{array}{c} \forall w^1 \in W^1, \forall \mu_h \in M_h, \\ \left(\gamma_\Gamma^2\left(\Pi_h^{21} w^1\right) + \gamma_\Gamma^1\left(w^1 - \Pi_h^{11} w^1\right), \mu_h\right)_{1/2,\Gamma} = 0 \end{array} \tag{35}$$

Proof. We define M_h, F and α by

$$\begin{gathered} M_h \in \mathcal{M}_{NSG}(\mathbb{R}), \\ M_h = (\eta_i, \eta_j)_{1/2,\Gamma}, \quad 1 \leq i, j \leq NSG, \end{gathered}$$

$$\begin{gathered} F : H^1\left(\Omega^F\right) \to \mathbb{R}^{NSG}, \\ F(w) = (F_1(w), \ldots, F_{NSG}(w))^t, \\ F_i(w) = \left(\gamma_\Gamma^1\left(-w + \Pi_h^{11} w\right), \eta_i\right)_{1/2,\Gamma} \end{gathered}$$

and

$$\begin{gathered} \alpha : H^1\left(\Omega^F\right) \to \mathbb{R}^{NSG}, \\ \alpha(w) = (\alpha_1(w), \ldots, \alpha_{NSG}(w))^t, \\ \alpha_i(w) = M_h^{-1} F(w). \end{gathered}$$

We set

$$\begin{gathered} \pi_h^{21} : H^1\left(\Omega^F\right) \to \Psi_h, \\ \pi_h^{21} w = \sum_{i=1}^{NSG} \alpha_i(w)\, \psi_{s(i)} \end{gathered}$$

where $s : \{1, \ldots, NSG\} \to \{1, \ldots, NSS\}$ is the application given by (32).

Let us define

$$\begin{gathered} \Pi_h^{21} : W^1 \to W_h^2 \\ \Pi_h^{21} w^1 = \left(\pi_h^{21} w^{1,1}, \pi_h^{21} w^{1,2}, \pi_h^{21} w^{1,3}\right) \end{gathered}$$

where $w^1 = \left(w^{1,1}, w^{1,2}, w^{1,3}\right)$ is an element of W^1.

Since the application trace and the operator Π_h^{11} are linear and continuous, the operator Π_h^{21} has the same properties.

For $w \in H^1\left(\Omega^F\right)$ we have

$$\pi_h^{21} w = \sum_{i=1}^{NSG} \alpha_i(w)\, \psi_{s(i)}$$

and consequently

$$\gamma_\Gamma^2\left(\pi_h^{21} w\right) = \sum_{i=1}^{NSG} \alpha_i(w)\, \psi_{s(i)} \mid_\Gamma .$$

From (32) we obtain

$$\gamma_\Gamma^2\left(\pi_h^{21} w\right) = \sum_{i=1}^{NSG} \alpha_i(w)\, \eta_i$$

and it follows that

$$\left(\gamma_\Gamma^2\left(\pi_h^{21} w\right), \eta_j\right)_{1/2,\Gamma} = \left(\sum_{i=1}^{NSG} \alpha_i(w)\, \eta_i, \eta_j\right)_{1/2,\Gamma}, \quad \forall j = 1, \ldots, NSG.$$

If we write this equality in matrix form and use that M_h is symmetrical, we have

$$\begin{pmatrix} \left(\gamma_\Gamma^2\left(\pi_h^{21}w\right),\eta_1\right)_{1/2,\Gamma} \\ \vdots \\ \left(\gamma_\Gamma^2\left(\pi_h^{21}w\right),\eta_{NSG}\right)_{1/2,\Gamma} \end{pmatrix} = M_h^t\alpha(w) = M_h M_h^{-1}F(w) = F(w)$$

and the equality (35) is proved. $\square$

Proposition 3 *There is a linear operator Π_h^{22} from W^2 to W_h^2 and a positive constant $c_{22}(h)$ depending on h, such that*

$$\forall w^2 \in W^2, \left\|\Pi_h^{22}w^2\right\|_{W^2} \le c_{22}(h)\left\|w^2\right\|_{W^2} \tag{36}$$

and

$$\begin{gathered} \forall w^2 \in W^2, \forall \mu_h \in M_h, \\ \left(\gamma_\Gamma^2\left(w^2 - \Pi_h^{22}w^2\right),\mu_h\right)_{1/2,\Gamma} = 0. \end{gathered} \tag{37}$$

Proof. We define G and β by

$$\begin{gathered} G:\left\{w\in H^1\left(\Omega^F\right); w\mid_{\Sigma^1}=0\right\} \to \mathbb{R}^{NSG}, \\ G(w) = \left(G_1(w),\ldots,G_{NSG}(w)\right)^t, \\ G_i(w) = \left(\gamma_\Gamma^2(w),\eta_i\right)_{1/2,\Gamma} \end{gathered}$$

and

$$\begin{gathered} \beta:\left\{w\in H^1\left(\Omega^F\right); w\mid_{\Sigma^1}=0\right\} \to \mathbb{R}^{NSG}, \\ \beta(w) = \left(\beta_1(w),\ldots,\beta_{NSG}(w)\right)^t, \\ \beta_i(w) = M_h^{-1}G(w). \end{gathered}$$

We set

$$\begin{gathered} \pi_h^{22}:\left\{w\in H^1\left(\Omega^F\right); w\mid_{\Sigma^1}=0\right\} \to \Psi_h, \\ \pi_h^{22}w = \sum_{i=1}^{NSG}\beta_i(w)\,\psi_{s(i)} \end{gathered}$$

where $s:\{1,\ldots,NSG\}\to\{1,\ldots,NSS\}$ is the application given by (32).

Let us define

$$\begin{gathered} \Pi_h^{22}:W^2\to W_h^2 \\ \Pi_h^{22}w^2 = \left(\pi_h^{22}w^{2,1},\pi_h^{22}w^{2,2},\pi_h^{22}w^{2,3}\right) \end{gathered}$$

where $w^2=\left(w^{2,1},w^{2,2},w^{2,3}\right)$ is an element of W^2 .

According to the definition of G, β, π_h^{22} and using the fact that the trace operator γ_Γ^2 is linear and continuous, we obtain that the operator Π_h^{22} is linear and continuous.

For $w\in\left\{w\in H^1\left(\Omega^F\right); w\mid_{\Sigma^1}=0\right\}$, we have

$$\pi_h^{22}w = \sum_{i=1}^{NSG}\beta_i(w)\,\psi_{s(i)},$$

hence

$$\gamma_\Gamma^2 \left(\pi_h^{22} w\right) = \sum_{i=1}^{NSG} \beta_i\left(w\right)\psi_{s(i)}\mid_\Gamma .$$

From (32) we obtain

$$\gamma_\Gamma^2\left(\pi_h^{21} w\right) = \sum_{i=1}^{NSG} \beta_i\left(w\right)\eta_i$$

and infer that

$$\left(\gamma_\Gamma^2\left(\pi_h^{22} w\right), \eta_j\right)_{1/2,\Gamma} = \left(\sum_{i=1}^{NSG} \beta_i\left(w\right)\eta_i, \eta_j\right)_{1/2,\Gamma}, \quad \forall j = 1, \ldots, NSG.$$

Writing this equality in matrix form and using that M_h is symmetrical, we deduce that

$$\begin{pmatrix} \left(\gamma_\Gamma^2\left(\pi_h^{22} w\right), \eta_1\right)_{1/2,\Gamma} \\ \vdots \\ \left(\gamma_\Gamma^2\left(\pi_h^{22} w\right), \eta_{NSG}\right)_{1/2,\Gamma} \end{pmatrix} = M_h^t \beta\left(w\right) = M_h M_h^{-1} G\left(w\right) = \begin{pmatrix} \left(\gamma_\Gamma^2\left(w\right), \eta_1\right)_{1/2,\Gamma} \\ \vdots \\ \left(\gamma_\Gamma^2\left(w\right), \eta_{NSG}\right)_{1/2,\Gamma} \end{pmatrix}$$

and the equality (37) is proved. □

We conclude from the Propositions 1, 2 and 3 that the interpolation operator Π_h verifies the conditions (23) and (24), where the constant c depends on h and consequently the discrete version of the problem (22) is well posed.

6 FINAL REMARKS

In this paper we have presented an algorithm for solving a three dimensional fluid structure interaction problem. At each time step, we have to solve a mixed hybrid system. We have proved that the discrete problem is well posed by constructing an interpolation operator.

The error analysis follows a standard way as in the abstract framework for the mixed hybrid problem, see for example [11].

Numerical tests have been performed.

At each time step, we solve numericaly the mixed hybrid linear system using the algorithm presented in [15] (which is a Uzawa like algorithm), where the fluid and structure equations are solved separately at each iteration. The Lagrange multiplier method was used to treat the continuity of the velocity on the interface.

The meshes are obtained by using MODULEF [16]. The fluid mesh has 336 tetrahedrons and the structure mesh has 366 tetrahedrons. The meshes are matching on the interface.

The fluid is solved numericaly by the software NSP1B3 [17] and the structure by MODULEF.

The numerical tests performed by us show that the computed fluid and structure velocities are almost equal at the interface.

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