SENSITIVITY AND APPROXIMATION OF FLUID-STRUCTURE EQUATIONS COUPLED BY VIRTUAL CONTROL METHOD

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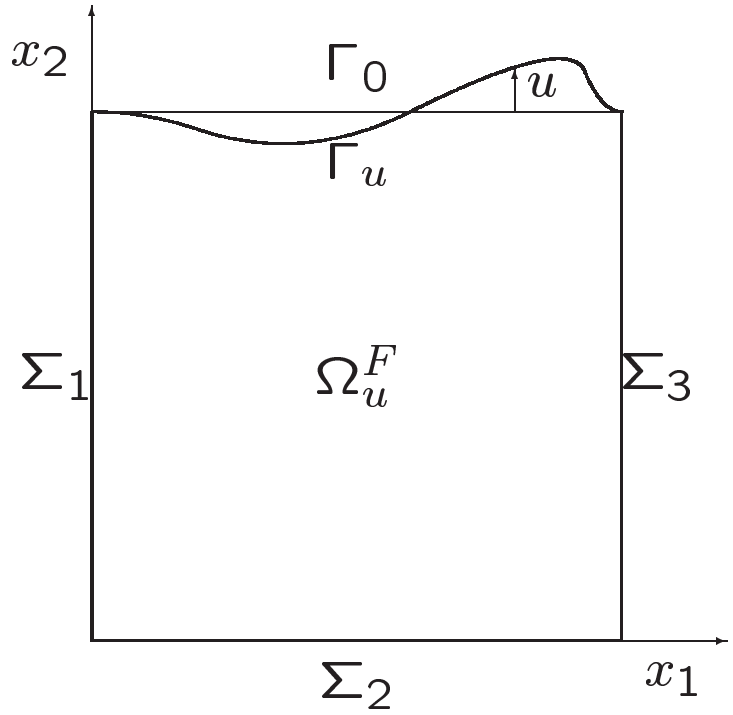


$$
\begin{aligned}
\Omega_u^F &= \left\{(x_1, x_2) \in \mathbb{R}^2;\ x_1 \in (0, L),\ 0 < x_2 < H + u\,(x_1)\right\},\\
\Gamma_u &= \left\{(x_1, x_2) \in \mathbb{R}^2;\ x_1 \in (0, L),\ x_2 = H + u\,(x_1)\right\},\\
\Sigma &= \Sigma_1 \cup \overline{\Sigma}_2 \cup \Sigma_3.
\end{aligned}
$$

Fluid-structure coupled equations

$$
\begin{aligned}
EI\, u''''(x_1) &= -\left(\sigma^F n \cdot e_2\right)_{(x_1, H+u(x_1))} \sqrt{1+\left(u'(x_1)\right)^2} + f^S(x_1) \\
u(0) &= u(L) = u'(0) = u'(L) = 0 \\
\int_0^L u(x_1)\, dx_1 &= 0 \\
e &\leq \inf_{x_1 \in [0,L]} \{H + u(x_1)\} \\
-\mu \Delta v + \nabla p &= f^F, \quad in\ \Omega_u^F \\
\text{div}\ v &= 0, \quad in\ \Omega_u^F \\
v &= g, \quad on\ \Sigma \\
v &= 0, \quad on\ \Gamma_u
\end{aligned}
$$

Control approach

Let $\widehat{\lambda} : (0, L) \to \mathbb{R}$ be the control function.

Solve the structure equations:

$$\begin{aligned} EI\, u''''(x_1) &= -\widehat{\lambda}(x_1) + f^S(x_1), \quad \forall x_1 \in (0, L) \\ u(0) &= u(L) = u'(0) = u'(L) = 0 \\ \int_0^L u(x_1)\, dx_1 &= 0 \end{aligned}$$

Solve the Stokes equations with boundary conditions: $v = g$ on Σ, $v_1 = 0$ on Γ_u and

$$\left(\sigma^F n \cdot e_2\right)_{(x_1, H+u(x_1))} = \frac{\widehat{\lambda}(x_1)}{\sqrt{1 + (u'(x_1))^2}}, \quad \forall x_1 \in (0, L).$$

The control problem is to find $\widehat{\lambda}$, such that $v_2 = 0$ on Γ_u.

Regularity of the control

If the system of fluid-structure equations has a strong solution $u \in H^4(0,L)$, $v \in \left(H^2\left(\Omega_u^F\right)\right)^2$ and $p \in H^1\left(\Omega_u^F\right)$, then

$$\widehat{\lambda}(x_1) = \left(\sigma^F n \cdot e_2\right)_{(x_1, H+u(x_1))} \sqrt{1+\left(u'(x_1)\right)^2} \in L^2(0,L).$$

In fact, the control is even smoother. The initial system is equivalent to the controllability problem.

If the system of fluid-structure equations has only a week solution $u \in H^2(0,L)$, $v \in \left(H^1\left(\Omega_u^F\right)\right)^2$ and $p \in L^2\left(\Omega_u^F\right)$, then $\widehat{\lambda}$ is well defined in a space like the dual of $H_{00}^{1/2}(0,L)$, which is larger than $L^2(0,L)$. In this case, we have only $\inf \frac{1}{2}\left\|v_{2|\Gamma_u}\right\|^2 = 0$ for $\widehat{\lambda} \in L^2(0,L)$.

Weak formulation for the structure equations

We set $U = H_0^2(0, L)$, $a_S : U \times U \to \mathbb{R}$,

$$a_S(\phi, \psi) = EI \int_0^L \phi''(x_1)\, \psi''(x_1)\, dx_1 .$$

Proposition 1 *Let $f^S \in U'$ and $\eta \in L^2(0, L)$. Then, the problem: Find $u \in U$ such that*

$$a_S(u, \psi) = \int_0^L \eta(x_1)\, \psi(x_1)\, dx_1 + \left\langle f^S, \psi \right\rangle_{U',U} \qquad \forall \psi \in U$$

has a unique solution. Moreover the solution $u \in \mathcal{C}^1([0, L])$ and we have the $L^\infty(0, L)$ estimate:

$$\|u\|_{L^\infty(0,L)} \leq C_1 \|\eta\|_{L^2(0,L)} + C_2 \left\|f^S\right\|_{U'}$$

where C_1 and C_2 are constants.

Obtaining constant volume of the fluid domain

We denote by $L_0^2(0,L) = \left\{\eta \in L^2(0,L);\ \int_0^L \eta(x_1)\,dx_1 = 0\right\}$.

Proposition 2 *Let $f^S \in U'$ and $\eta \in L_0^2(0,L)$.*

i) Then there exist an unique $u \in U$, such that $\int_0^L u(x_1)\,dx_1 = 0$ and an unique constant $c \in \mathbb{R}$ solutions of

$$a_S(u,\psi) = \int_0^L (\eta(x_1) + c)\,\psi(x_1)\,dx_1 + \left\langle f^S, \psi \right\rangle_{U',U} \quad \forall \psi \in U$$

ii) Let $u_0 \in U$, such that $\int_0^L u_0\,dx_1 = 0$ and $c_0 \in \mathbb{R}$ solutions of

$$a_S(u_0,\psi) = c_0 \int_0^L \psi(x_1)\,dx_1 + \left\langle f^S, \psi \right\rangle_{U',U} \quad \forall \psi \in U$$

and $u_\eta \in U$, *such that* $\int_0^L u_\eta \, dx_1 = 0$ *and* $\ell(\eta) \in \mathbb{R}$ *solutions of*

$$a_S(u_\eta, \psi) = \int_0^L (\eta(x_1) + \ell(\eta)) \, \psi(x_1) \, dx_1 \quad \forall \psi \in U.$$

Then, $u = u_0 + u_\eta$, $c = c_0 + \ell(\eta)$ *and the applications*

$$\eta \in L^2_0(0, L) \mapsto u_\eta \in U, \qquad \eta \in L^2_0(0, L) \mapsto \ell(\eta) \in \mathbb{R}$$

are linear and continuous.

In order to obtain a fluid domain with constant volume, we have to take the control in the image of the application

$$\eta \in L^2_0(0, L) \mapsto \eta + c_0 + \ell(\eta) \in L^2(0, L)$$

Mixed formulation in variable fluid domain

We denote by $\mu > 0$ the viscosity of the fluid and by $\epsilon(v) = \left(\epsilon_{ij}(v)\right)_{1\leq i,j\leq 2}$ the deformation rate tensor, where $\epsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$. We set

$$\begin{aligned} W_u &= \left\{ w \in \left(H^1\left(\Omega_u^F\right)\right)^2;\ w_1 = 0 \text{ on } \partial\Omega_u^F, w_2 = 0 \text{ on } \overline{\Sigma} \right\}, \\ Q_u &= L^2\left(\Omega_u^F\right). \end{aligned}$$

$$a_F(u, v, w) = 2\mu \sum_{i,j=1}^{2} \int_{\Omega_u^F} \epsilon_{ij}(v)\epsilon_{ij}(w)\, dx$$

$$b_F(u, w, q) = -\int_{\Omega_u^F} (\text{div } w)\, q\, dx.$$

Proposition 3 *For all u in $\mathcal{U}_{ad}$ and λ in $L^2(\Gamma_u)$, the problem:*
Find $(v,\ p) \in W_u \times Q_u$ such that

$$\begin{cases} a_F(u,v,w) + b_F(u,w,p) &= \displaystyle\sum_{i=1}^{2} \left(\int_{\Omega_u^F} f_i^F w_i \, dx \right) + \int_{\Gamma_u} \lambda w_2 \, d\sigma, & \forall w \in W_u \\ b_F(u,v,q) &= 0, & \forall q \in Q_u \end{cases}$$

has a unique solution.

$$\begin{cases} -\mu \Delta v + \nabla p &= f^F & \text{in} & \Omega_u^F \\ \text{div } v &= 0 & \text{in} & \Omega_u^F \\ v &= 0 & \text{on} & \Sigma \\ \left(\sigma^F n\right) \cdot e_2 &= \lambda & \text{on} & \Gamma_u \\ v_1 &= 0 & \text{on} & \Gamma_u \end{cases}$$

ALE transformation

$$T_u : \overline{\Omega_0^F} \to \overline{\Omega_u^F} \qquad T_u\left(\widehat{x}_1, \widehat{x}_2\right) = \left(\widehat{x}_1, \frac{H+u(\widehat{x}_1)}{H}\widehat{x}_2\right)$$

$$T_u^{-1} : \overline{\Omega_u^F} \to \overline{\Omega_0^F} \qquad T_u^{-1}\left(x_1, x_2\right) = \left(x_1, \frac{Hx_2}{H+u(x_1)}\right)$$

$$\nabla T_u\left(\widehat{x}\right) = \begin{pmatrix} 1 & 0 \\ \frac{u'(\widehat{x}_1)}{H}\widehat{x}_2 & \frac{H+u(\widehat{x}_1)}{H} \end{pmatrix}$$

$$\nabla\left(T_u^{-1}\right)(x) = \begin{pmatrix} 1 & 0 \\ \frac{-u'(x_1)Hx_2}{\left(H+u(x_1)\right)^2} & \frac{H}{H+u(x_1)} \end{pmatrix}$$

$$\left(\nabla T_u\right)^{-1}\left(\widehat{x}\right) = \nabla\left(T_u^{-1}\right)(x) = \nabla\left(T_u^{-1}\right)\left(T_u(\widehat{x})\right)$$

Mixed formulation in a fixed fluid domain. Notations

$$\begin{aligned}
\widehat{a}_F(u,\widehat{v},\widehat{w}) &= 2\mu \sum_{i,j,k,\ell=1}^{2} \int_{\Omega_0^F} a_{k,\ell}^{i,j}(u,\widehat{x}) \frac{\partial \widehat{v}_i}{\partial \widehat{x}_k} \frac{\partial \widehat{w}_j}{\partial \widehat{x}_\ell} \, d\widehat{x} \\
\widehat{b}_F(u,\widehat{w},\widehat{q}) &= -\int_{\Omega_0^F} \left(\frac{\partial \widehat{v}_1}{\partial \widehat{x}_1} \frac{H + u(\widehat{x}_1)}{H} - \frac{\partial \widehat{v}_1}{\partial \widehat{x}_2} \frac{u'(\widehat{x}_1)\widehat{x}_2}{H} + \frac{\partial \widehat{v}_2}{\partial \widehat{x}_2} \right) \widehat{q} \, d\widehat{x}.
\end{aligned}$$

$$\left\langle \widehat{f}^F(u), \widehat{w} \right\rangle_{\widehat{W}',\widehat{W}} = \sum_{i=1}^{2} \int_{\Omega_0^F} f_i^F \widehat{w}_i \det\left(\nabla T_u(\widehat{x})\right) d\widehat{x}$$

Mixed formulation in a fixed fluid domain. Existence

$$\begin{aligned} \widehat{W} &= \left\{ \widehat{w} \in \left(H^1\left(\Omega_0^F\right)\right)^2;\ \ \widehat{w}_1 = 0 \text{ on } \partial\Omega_0^F, \widehat{w}_2 = 0 \text{ on } \overline{\Sigma} \right\} \\ \widehat{Q} &= L^2\left(\Omega_0^F\right) \end{aligned}$$

Proposition 4 *For all u in $\mathcal{U}_{ad}$ and $\widehat{\lambda}$ in $L^2(\Gamma_0)$, the problem: Find $(\widehat{v},\ \widehat{p}) \in \widehat{W} \times \widehat{Q}$ such that*

$$\begin{cases} \widehat{a}_F(u, \widehat{v}, \widehat{w}) + \widehat{b}_F(u, \widehat{w}, \widehat{p}) &= \left\langle \widehat{f}^F(u), \widehat{w} \right\rangle + \displaystyle\int_{\Gamma_0} \widehat{\lambda}\widehat{w}_2 \, d\widehat{\sigma}, & \forall \widehat{w} \in \widehat{W} \\ \widehat{b}_F(u, \widehat{v}, \widehat{q}) &= 0, & \forall \widehat{q} \in \widehat{Q} \end{cases}$$

has a unique solution.

Optimal control setting

$$\inf j\left(\widehat{\lambda}\right) = \frac{1}{2}\left\|\widehat{v}_2\right\|^2_{0,\Gamma_0}$$

1) $\widehat{\lambda} \in \mathcal{F}_{ad}$

2) $u\left(\widehat{\lambda}\right) \in \mathcal{U}_{ad}$ such that

$$A_S\, u\left(\widehat{\lambda}\right) = -C_S\, \widehat{\lambda} + f^S$$

3) $\widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \in \widehat{W}, \quad \widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \in \widehat{Q}$ such that

$$\begin{pmatrix} A_F\left(u\left(\widehat{\lambda}\right)\right) & B_F^*\left(u\left(\widehat{\lambda}\right)\right) \\ B_F\left(u\left(\widehat{\lambda}\right)\right) & 0 \end{pmatrix} \begin{pmatrix} \widehat{v}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \\ \widehat{p}\left(u\left(\widehat{\lambda}\right), \widehat{\lambda}\right) \end{pmatrix} = \begin{pmatrix} \widehat{f}^F\left(u\right) + C_F \widehat{\lambda} \\ 0 \end{pmatrix}$$

Sensitivity

Theorem 1 *The cost function j is Fréchet differentiable and we have*

$$\begin{aligned} j'\left(\widehat{\lambda}\right)\widehat{\mu} \;=\; & \left(\frac{\partial \widehat{a}_F}{\partial u}\left(u,\widehat{v},\widehat{z}\right)+\frac{\partial \widehat{b}_F}{\partial u}\left(u,\widehat{z},\widehat{p}\right)+\frac{\partial \widehat{b}_F}{\partial u}\left(u,\widehat{v},\widehat{r}\right)\right) A_S^{-1} C_S \widehat{\mu} \\ & -\int_{\Omega_0^F} \frac{\left(A_S^{-1} C_S \widehat{\mu}\right)(\widehat{x}_1)}{H} f^F \cdot \widehat{z}\, d\widehat{x} + \int_{\Gamma_0} \widehat{v}_2 \frac{\partial \widehat{v}_2}{\partial \widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu}\, d\widehat{\sigma}, \end{aligned}$$

where the displacement u is computed from

$$A_S\, u = -C_S\, \widehat{\lambda} + f^S,$$

the velocity $\widehat{v}$ *and the pressure* $\widehat{p}$ *of the fluid are computed from*

$$\begin{cases} \widehat{a}_F\left(u,\widehat{v},\widehat{w}\right)+\widehat{b}_F\left(u,\widehat{w},\widehat{p}\right) &= \left\langle \widehat{f}^F\left(u\right),\widehat{w}\right\rangle+\int_{\Gamma_0}\widehat{\lambda}\widehat{w}_2\,d\widehat{\sigma}, \quad \forall\widehat{w}\in\widehat{W} \\ \widehat{b}_F\left(u,\widehat{v},\widehat{q}\right) &= 0, \quad \forall\widehat{q}\in\widehat{Q}, \end{cases} \tag{1}$$

the adjoint state $\widehat{z}$ *and* $\widehat{r}$ *are computed from (1) with* $\int_{\Gamma_0}\widehat{v}_2\widehat{w}_2\,d\widehat{\sigma}$ *as the right side and* $\frac{\partial\widehat{v}_2}{\partial\widehat{\lambda}}\left(u,\widehat{\lambda}\right)\widehat{\mu}$ *is computed from (1) with* $\int_{\Gamma_0}\widehat{\mu}\widehat{w}_2\,d\widehat{\sigma}$ *as the right side.*

Approximation

For $1 \leq i \leq m$, $\phi_i \in L^2(0, L)$ given functions and $\alpha_i \in \mathbb{R}$ discret controls to be identified.

There exist $u_0 \in U$ such that $\int_0^L u_0 \, dx_1 = 0$ and $c_0 \in \mathbb{R}$ solutions of the structure equations with $f^S = 0$.

There exist $u_i \in U$ such that $\int_0^L u_i \, dx_1 = 0$ and $c_i \in \mathbb{R}$ solutions of

$$a_S(u_i, \psi) = \int_0^L (\phi_i(x_1) + c_i) \psi(x_1) \, dx_1 \quad \forall \psi \in U.$$

We take $\widehat{\lambda}(x_1, H) = -c_0 + \sum_{i=1}^m \alpha_i (-\phi_i(x_1) - c_i)$ and we obtain the displacement $u = u_0 + \sum_{i=1}^m \alpha_i u_i$ such that $\int_0^L u \, dx_1 = 0$.

Minimization problem without constraints

$$\mathcal{J}(\alpha_1, \ldots, \alpha_m) = j\left(-c_0 + \sum_{i=1}^{m} \alpha_i \left(-\phi_i(x_1) - c_i\right)\right)$$

$$\frac{\partial \mathcal{J}}{\partial \alpha_k}(\alpha_1, \ldots, \alpha_m) = j'\left(\widehat{\lambda} = -c_0 + \sum_{i=1}^{m} \alpha_i \left(-\phi_i(\widehat{x}_1) - c_i\right)\right)(-\phi_k - c_k)$$

We ignore for instant the constraint $e \leq \inf_{x_1 \in [0,L]} \{H + u(x_1)\}$ and we can use the quasi-Newton method like Broyden, Fletcher, Goldfarb, Shanno (BFGS) algorithm for the minimization problem without constraints

$$\inf \mathcal{J}(\alpha_1, \ldots, \alpha_m).$$

Numerical tests

We are interested in simulating the blood flow through arteries.

Geometry: $L = 3\ cm,\ H = 0.5\ cm.$

Vessel: $h = 0.1\ cm,\ E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2},\ \rho^S = 1.1\ \frac{g}{cm^3},\ g_0 = 981\ \frac{cm}{s^2},$
$f^S(x_1) = -g_0 \rho^S h$

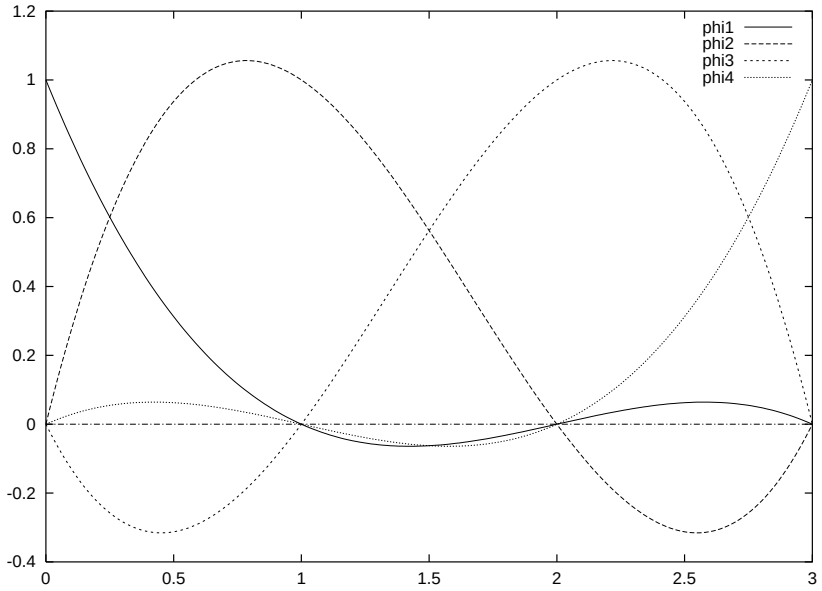
Blood: $\mu = 0.035\ \frac{g}{cm \cdot s},\ \rho^F = 1\ \frac{g}{cm^3},$

$$\begin{aligned} v_1(x_1, x_2) &= \begin{cases} \left(1 - \frac{x_2^2}{H^2}\right) V_0, & (x_1, x_2) \in \Sigma_1 \cup \Sigma_3 \\ V_0, & (x_1, x_2) \in \Sigma_2 \end{cases} \\ v_2(x_1, x_2) &= 0, \quad (x_1, x_2) \in \Sigma \end{aligned}$$

where $V_0 = 30\ \frac{cm}{s}$. The volume force in fluid is $\mathbf{f}^F = (0, -g_0 \rho^F)^T$.

The shape functions for the approximation of the control

We take $m = 4$. Let $\xi_i = (i-1)L/(m-1)$ for $1 \leq i \leq m$. There exist ϕ_i polynomial functions of degree 3, such that $\phi_i(\xi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker's symbol.



Solving the structure equations

From the regularity of ϕ_i, we can use the the following strong formulation in order to compute u_i, c_i:

$$\begin{aligned} EI\, u_i''''(x_1) &= \phi_i(x_1) + c_i,\ \forall x_1 \in (0, L) \\ u_i(0) &= u_i'(0) = 0, \\ u_i(L) &= u_i'(L) = 0 \\ \int_0^L u_i(x_1)\, dx_1 &= 0. \end{aligned}$$

We have computed u_i, c_i exactly, using the software *Mathematica*.

The displacements u_i are polynomial functions of degree 7. We can use finit element like shape functions for ϕ_i, but in this case we have to use the weak formulation in order to compute u_i and c_i.

Solving by the BFGS algorithm

For the fluid we have used the Mixed Finite Element Method, $P2$ Lagrange triangles for the velocity and $P1$ for the pressure.

The numerical tests have been produced using *freefem++ v1.27*. We have used the BFGS algorithm for the minimization problem with the starting point $\alpha = 0$.

Iterations	$\mathcal{J}$
0	8.361636438
1	7.699302111
2	0.1545378956
3	0.1516218053
4	0.1491254925
5	0.1490809944

After 5 iterations we have obtained

$$(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (13.798,\ 2.812,\ -2.645,\ -13.965)$$

and the gradient of the cost function was

$$\nabla \mathcal{J} = (-7.937e-05,\ 0.0010,\ -0.0011,\ 0.0001)^T.$$

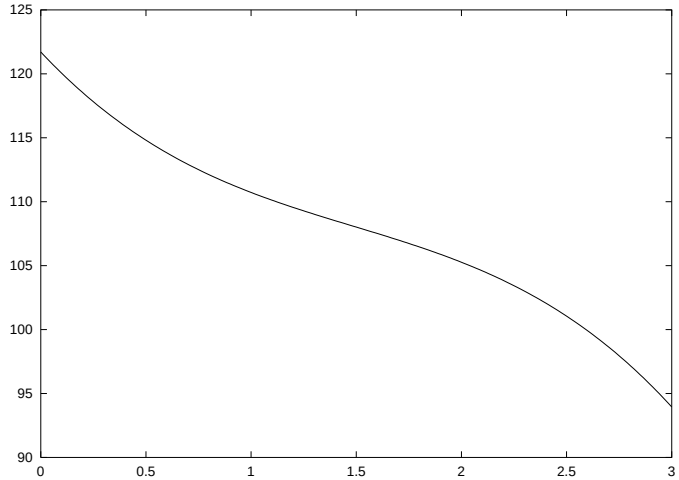
The condition $e \leq \inf_{x_1 \in [0,L]} \{H + u(x_1)\}$ was not violated.

Analytic gradient versus Finite Differences Method

In order to compute $\nabla \mathcal{J}(\alpha)$, we have to solve the adjoint state problem and m linear systems which have the same matrix. The linear systems were solved by LU decomposition. So when we compute $\nabla \mathcal{J}(\alpha)$ we can use the same LU decomposition obtained computing $\mathcal{J}(\alpha)$.

If we compute $\nabla \mathcal{J}(\alpha)$ by the Finite Differences Method, we have to solve m linear systems, but the matrices are different because u changes, so using the analytic formula of the gradient is more advantageous.

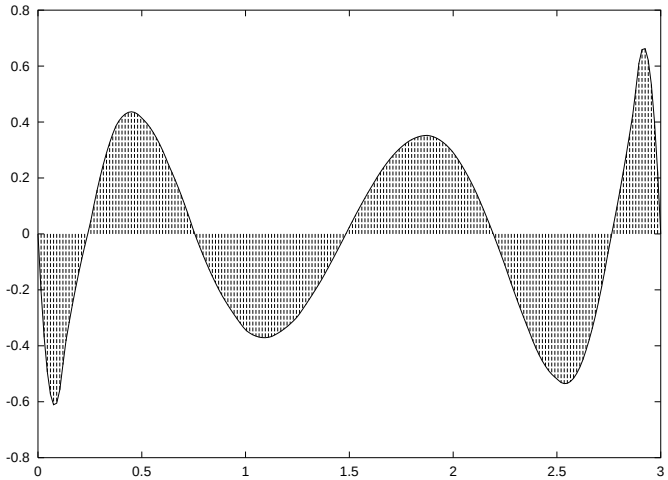
The optimal control



The applied stresses $[dyn/cm^2]$ on the interface: $-\widehat{\lambda}$

We have obtained $\left\|\widehat{\lambda}+\widehat{p}\right\|^2_{0,\Gamma_0} = 0.01725973274$, in other words, the vertical component of the stresses exerced by the fluid on the interface depends on the pressure only, $-\widehat{\lambda} \approx \widehat{p}_{|\Gamma_0}$.

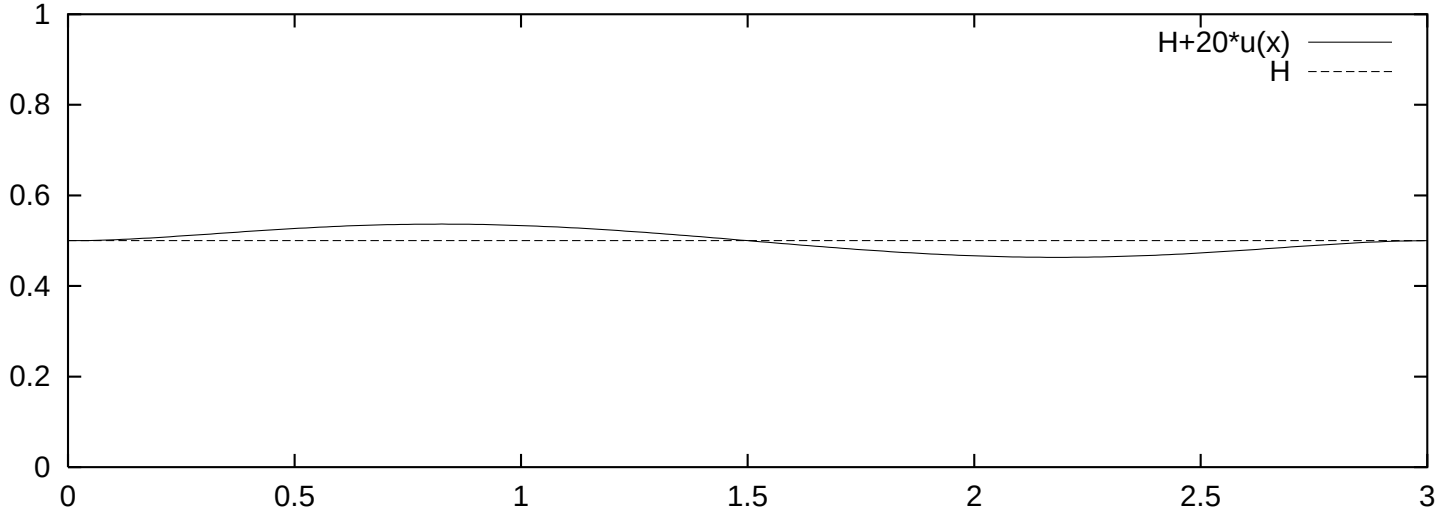
The optimal observation



The velocity $[cm/s]$ on the boundary Γ_0: $\widehat{v}$

The velocity on the boundary Γ_0 is not constant, but the maximum of the absolute value is less than 0.7 cm/s.

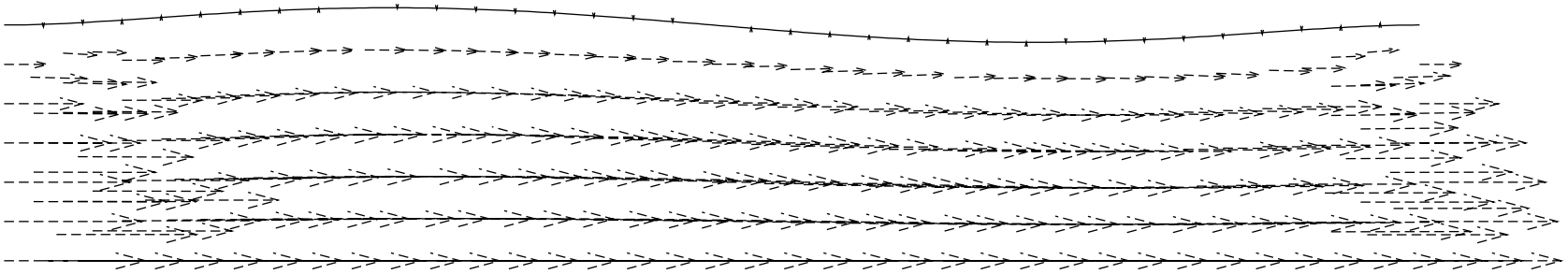
The displacement of the vessel



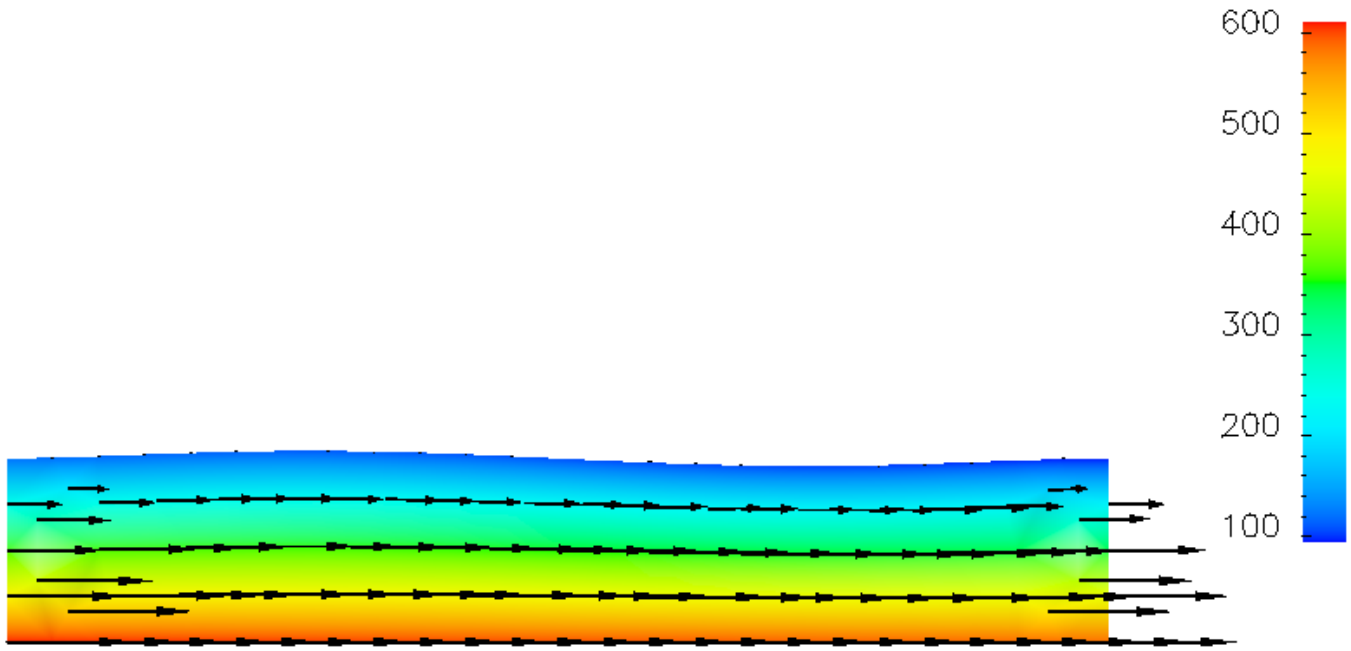
The displacement $[cm]$ of the vessel magnified by a factor 20

The displacement of the vessel is very small, it is less than 0.04 cm.

The velocity of the blood



The velocity $[cm/s]$ reduced by a factor 100



pression and velocity

Conclusions

Optimal control setting, analytic formula of the gradient

Satisfactory numerical results ($\inf j \approx 0.14$) for a few number of controls ($m = 4$)

Efficiency: only 5 iterations of the BFGS algorithm and we can use the same matrix factorization for $\mathcal{J}(\alpha)$ and $\nabla\mathcal{J}(\alpha)$

Future works

Imposing at the inflow and outflow: $p = p_0$ and $v \times n = 0$

Unsteady case

3D