

Augmented Lagrangian Method for fluid-structure interaction

Cornel Marius Murea

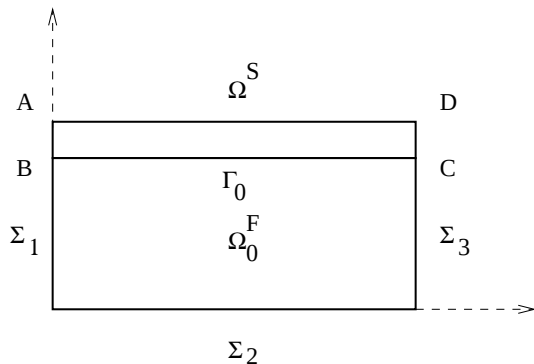
Laboratoire de Mathématiques, Informatique et Applications,
Université de Haute-Alsace, France

e-mail: cornel.murea@uha.fr

<http://www.lmia.uha.fr/~edp/murea/>

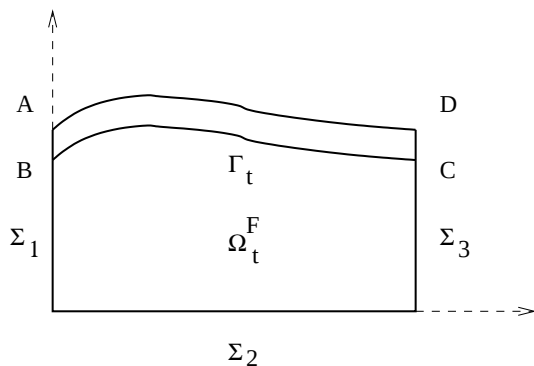
Mulhouse, Decembre 8, 2006

Initial geometrical configuration



$$\partial\Omega_0^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_0.$$

Intermediate geometrical configuration



$$\partial\Omega_t^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_t.$$

Linear elasticity equations

Find the displacement $\mathbf{u} = (u_1, u_2)^T : \Omega^S \times [0, T] \rightarrow \mathbb{R}^2$ of the structure such that

$$\begin{aligned}\rho^S \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \sigma^S &= \mathbf{f}^S, \quad \text{in } \Omega^S \times (0, T) \\ \sigma^S &= \lambda^S (\nabla \cdot \mathbf{u}) \mathbb{I}_2 + 2\mu^S \epsilon(\mathbf{u}) \\ \epsilon(\mathbf{u}) &= \frac{1}{2} \left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T \right) \\ \mathbf{u} &= 0, \quad \text{on } \Gamma_D \times (0, T) \\ \sigma^S \mathbf{n}^S &= 0, \quad \text{on } \Gamma_N \times (0, T)\end{aligned}$$

$$\Gamma_D = [AB] \cup [CD], \quad \Gamma_N = [DA].$$

Navier-Stokes equations

Find the velocity $\mathbf{v}$ and the pressure p of the fluid such that

$$\rho^F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) - \nabla \cdot \sigma^F = \mathbf{f}^F, \quad \forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F$$

$$\nabla \cdot \mathbf{v} = 0, \quad \forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F$$

$$\sigma^F = -p \mathbb{I}_2 + 2\mu^F \epsilon(\mathbf{v})$$

$$\sigma^F \mathbf{n}^F = \mathbf{h}_{in}, \quad \text{on } \Sigma_1 \times (0, T)$$

$$\sigma^F \mathbf{n}^F = \mathbf{h}_{out}, \quad \text{on } \Sigma_3 \times (0, T)$$

$$\mathbf{v} \cdot \mathbf{n}^F = 0, \quad \text{on } \Sigma_2 \times (0, T)$$

$$\boldsymbol{\tau}^F \cdot (\sigma^F \mathbf{n}^F) = 0, \quad \text{on } \Sigma_2 \times (0, T)$$

Interface and initial conditions

The interface Γ_t is the image of the boundary Γ_0 by the map

$$\mathbb{T}(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t).$$

Interface conditions

$$\mathbf{v}(\mathbf{X} + \mathbf{u}(\mathbf{X}, t), t) = \frac{\partial \mathbf{u}}{\partial t}(\mathbf{X}, t), \quad \forall (\mathbf{X}, t) \in \Gamma_0 \times (0, T)$$

$$\left(\sigma^F \mathbf{n}^F \right)_{(\mathbf{X} + \mathbf{u}(\mathbf{X}, t), t)} \omega(\mathbf{X}, t) = - \left(\sigma^S \mathbf{n}^S \right)_{(\mathbf{X}, t)}, \quad \forall (\mathbf{X}, t) \in \Gamma_0 \times (0, T)$$

where $\omega(\mathbf{X}, t) = \|\text{cof}(\nabla \mathbb{T}) \mathbf{n}^S\|_{\mathbb{R}^2}$.

Initial conditions

$$\begin{aligned} \mathbf{u}(\mathbf{X}, t = 0) &= \mathbf{u}^0(\mathbf{X}), \quad \text{in } \Omega^S \\ \frac{\partial \mathbf{u}}{\partial t}(\mathbf{X}, t = 0) &= \dot{\mathbf{u}}^0(\mathbf{X}), \quad \text{in } \Omega^S \\ \mathbf{v}(\mathbf{x}, t = 0) &= \mathbf{v}^0(\mathbf{x}), \quad \text{in } \Omega_0^F \end{aligned}$$

Weak formulation of the structure

Find the displacement $\mathbf{u}$ of the structure

$$\int_{\Omega^S} \rho^S \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \mathbf{w}^S + a_S(\mathbf{u}, \mathbf{w}^S) = \int_{\Omega^S} \mathbf{f}^S \cdot \mathbf{w}^S + \int_{\Gamma_0} (\sigma^S \mathbf{n}^S) \cdot \mathbf{w}^S$$

$$\forall \mathbf{w}^S = 0 \text{ on } \Gamma_D$$

$$a_S(\mathbf{u}, \mathbf{w}^S) = \int_{\Omega^S} \lambda^S (\nabla \cdot \mathbf{u}) (\nabla \cdot \mathbf{w}^S) + \int_{\Omega^S} 2\mu^S \epsilon(\mathbf{u}) : \epsilon(\mathbf{w}^S)$$

Arbitrary Lagrangian Eulerian (ALE) framework. Notations

Let $\widehat{\Omega}^F$ be a reference fixed domain.

Let $\mathcal{A}_t$, $t \in [0, T]$ be a family of transformations such that

$$\begin{aligned}\mathcal{A}_t(\widehat{\mathbf{x}}) &= \mathbf{x}, \quad \forall \widehat{\mathbf{x}} \in \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \\ \mathcal{A}_t(\Gamma_0) &= \Gamma_t \\ \mathcal{A}_t(\widehat{\Omega}^F) &= \Omega_t^F\end{aligned}$$

$\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2)^T \in \widehat{\Omega}^F$ the ALE and $\mathbf{x} = (x_1, x_2)^T = \mathcal{A}_t(\widehat{\mathbf{x}})$ the Eulerian coordinates.

Let v be the velocity of the fluid in the Eulerian coordinates. The corresponding function in the ALE framework $\widehat{v} : \widehat{\Omega}^F \times [0, T] \rightarrow \mathbb{R}^2$ is defined by $\widehat{v}(\widehat{\mathbf{x}}, t) = v(\mathcal{A}_t(\widehat{\mathbf{x}}), t) = v(\mathbf{x}, t)$.

We denote the ALE time derivative by $\frac{\partial v}{\partial t}\big|_{\widehat{\mathbf{x}}}(\mathbf{x}, t) = \frac{\partial \widehat{v}}{\partial t}(\widehat{\mathbf{x}}, t)$ and the domain velocity by $\vartheta(\mathbf{x}, t) = \frac{\partial \mathcal{A}_t}{\partial t}(\widehat{\mathbf{x}})$.

Navier-Stokes equations in the ALE framework

Find the velocity $\mathbf{v}$ and the pressure p of the fluid such that

$$\begin{aligned}\rho^F \left(\frac{\partial \mathbf{v}}{\partial t} \Big|_{\hat{\mathbf{x}}} + ((\mathbf{v} - \boldsymbol{\vartheta}) \cdot \nabla) \mathbf{v} \right) - 2\mu^F \nabla \cdot \boldsymbol{\epsilon}(\mathbf{v}) + \nabla p &= \mathbf{f}^F, \\ \nabla \cdot \mathbf{v} &= 0,\end{aligned}$$

$$\forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F$$

$$\boldsymbol{\sigma}^F \mathbf{n}^F = \mathbf{h}_{in}, \quad \text{on } \Sigma_1 \times (0, T)$$

$$\boldsymbol{\sigma}^F \mathbf{n}^F = \mathbf{h}_{out}, \quad \text{on } \Sigma_3 \times (0, T)$$

$$\mathbf{v} \cdot \mathbf{n}^F = 0, \quad \text{on } \Sigma_2 \times (0, T)$$

$$\boldsymbol{\tau}^F \cdot (\boldsymbol{\sigma}^F \mathbf{n}^F) = 0, \quad \text{on } \Sigma_2 \times (0, T)$$

Weak formulation of the Navier-Stokes equations

Find $\mathbf{v} \cdot \mathbf{n}^F = 0$ on $\Sigma_2 \times (0, T)$ and p such that

$$\begin{aligned} & \int_{\Omega_t^F} \rho^F \frac{\partial \mathbf{v}}{\partial t} \Big|_{\hat{\mathbf{x}}} \cdot \mathbf{w}^F + \int_{\Omega_t^F} \rho^F (((\mathbf{v} - \vartheta) \cdot \nabla) \mathbf{v}) \cdot \mathbf{w}^F \\ & \quad + a_F(\mathbf{v}, \mathbf{w}^F) + b_F(\mathbf{w}^F, p) = \int_{\Omega_t^F} \mathbf{f}^F \cdot \mathbf{w}^F \\ & \quad + \int_{\Sigma_1} \mathbf{h}_{in} \cdot \mathbf{w}^F + \int_{\Sigma_3} \mathbf{h}_{out} \cdot \mathbf{w}^F + \int_{\Gamma_t} (\sigma^F \mathbf{n}^F) \cdot \mathbf{w}^F \end{aligned}$$

$\forall \mathbf{w} \cdot \mathbf{n}^F = 0$ on $\Sigma_2 \times (0, T)$ and

$$b_F(\mathbf{v}, q) = 0, \forall q$$

$$a_F(\mathbf{v}, \mathbf{w}^F) = \int_{\Omega_t^F} 2\mu^F \epsilon(\mathbf{v}) : \epsilon(\mathbf{w}^F)$$

$$b_F(\mathbf{w}^F, q) = - \int_{\Omega_t^F} (\nabla \cdot \mathbf{w}^F) q$$

Weak formulation of the fluid-structure interaction

Structure

$$\dots + \int_{\Gamma_0} \left(\sigma^S \mathbf{n}^S \right) \cdot \mathbf{w}^S, \quad \forall \mathbf{w}^S = 0 \text{ on } \Gamma_D \times (0, T)$$

Fluid

$$\dots + \int_{\Gamma_t} \left(\sigma^F \mathbf{n}^F \right) \cdot \mathbf{w}^F, \quad \forall \mathbf{w}^F \cdot \mathbf{n}^F = 0 \text{ on } \Sigma_2 \times (0, T)$$

$$b_F(\mathbf{v}, q) = 0, \quad \forall q$$

Interface conditions

$$\begin{aligned} \mathbf{v}(\mathbf{X} + \mathbf{u}(\mathbf{X}, t), t) &= \frac{\partial \mathbf{u}}{\partial t}(\mathbf{X}, t), \text{ on } \Gamma_0 \times (0, T) \\ \left(\sigma^F \mathbf{n}^F \right)_{(\mathbf{X} + \mathbf{u}(\mathbf{X}, t), t)} \omega(\mathbf{X}, t) &= - \left(\sigma^S \mathbf{n}^S \right)_{(\mathbf{X}, t)}, \text{ on } \Gamma_0 \times (0, T) \\ \mathbf{w}^F(\mathbf{X} + \mathbf{u}(\mathbf{X}, t), t) &= \mathbf{w}^S(\mathbf{X}, t), \text{ on } \Gamma_0 \times (0, T) \end{aligned}$$

Partitioned procedures by Lagrange multiplier

Find $\mathbf{u} = 0$ on Γ_D , $\mathbf{v} \cdot \mathbf{n}^F = 0$ on Σ_2 , p and η such that

$$\int_{\Omega^S} \rho^S \frac{\partial^2 \mathbf{u}}{\partial t^2} \cdot \mathbf{w}^S + a_S(\mathbf{u}, \mathbf{w}^S) - \int_{\Gamma_0} \hat{\boldsymbol{\eta}} \cdot \mathbf{w}^S \omega = \int_{\Omega^S} \mathbf{f}^S \cdot \mathbf{w}^S$$

$$\forall \mathbf{w}^S = 0 \text{ on } \Gamma_D$$

$$\begin{aligned} & \int_{\Omega_t^F} \rho^F \frac{\partial \mathbf{v}}{\partial t} \Big|_{\hat{\mathbf{x}}} \cdot \mathbf{w}^F + \int_{\Omega_t^F} \rho^F (((\mathbf{v} - \vartheta) \cdot \nabla) \mathbf{v}) \cdot \mathbf{w}^F \\ & + a_F(\mathbf{v}, \mathbf{w}^F) + b_F(\mathbf{w}^F, p) + \int_{\Gamma_t} \boldsymbol{\eta} \cdot \mathbf{w}^F \\ & = \int_{\Omega_t^F} \mathbf{f}^F \cdot \mathbf{w}^F + \int_{\Sigma_1} \mathbf{h}_{in} \cdot \mathbf{w}^F + \int_{\Sigma_3} \mathbf{h}_{out} \cdot \mathbf{w}^F \end{aligned}$$

$$\forall \mathbf{w}^F \cdot \mathbf{n}^F = 0 \text{ on } \Sigma_2$$

$$b_F(\mathbf{v}, q) = 0, \quad \forall q$$

$$\int_{\Gamma_0} \zeta \cdot \left(\hat{\mathbf{v}} - \frac{\partial \mathbf{u}}{\partial t} \right) \omega = 0, \quad \forall \zeta$$

Newmark algorithm

$\Delta t > 0$ the time step, $t_{n+1} = (n+1)\Delta t$,
 $\mathbf{u}^{n+1}$, $\dot{\mathbf{u}}^{n+1}$, $\ddot{\mathbf{u}}^{n+1}$ are approximations of
 $\mathbf{u}(\cdot, t_{n+1})$, $\frac{\partial \mathbf{u}}{\partial t}(\cdot, t_{n+1})$, $\frac{\partial^2 \mathbf{u}}{\partial t^2}(\cdot, t_{n+1})$ respectively.

Find $\mathbf{u}^{n+1}$, $\dot{\mathbf{u}}^{n+1}$, $\ddot{\mathbf{u}}^{n+1}$ such that

$$\int_{\Omega^S} \rho^S \ddot{\mathbf{u}}^{n+1} \cdot \mathbf{w}^S + a_S(\mathbf{u}^{n+1}, \mathbf{w}^S) - \int_{\Gamma_0} \hat{\boldsymbol{\eta}}^{n+1} \cdot \mathbf{w}^S \omega = \int_{\Omega^S} \mathbf{f}^S \cdot \mathbf{w}^S$$

$$\forall \mathbf{w}^S = 0 \text{ on } \Gamma_D$$

$$\dot{\mathbf{u}}^{n+1} = \dot{\mathbf{u}}^n + \Delta t ((1 - \delta) \ddot{\mathbf{u}}^n + \delta \ddot{\mathbf{u}}^{n+1})$$

$$\mathbf{u}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + (\Delta t)^2 \left(\left(\frac{1}{2} - \theta \right) \ddot{\mathbf{u}}^n + \theta \ddot{\mathbf{u}}^{n+1} \right)$$

1st order if $\delta \neq \frac{1}{2}$

2nd order if $\delta = \frac{1}{2}$ and $\theta \neq \frac{1}{12}$

4th order if $\delta = \frac{1}{2}$ and $\theta = \frac{1}{12}$

Implementation of Newmark algorithm: the v-form

Find $\dot{\mathbf{u}}^{n+1}$ such that

$$\mathcal{A}_S(\dot{\mathbf{u}}^{n+1}, \mathbf{w}^S) = \mathcal{L}_S(\mathbf{w}^S) + \int_{\Gamma_0} \hat{\boldsymbol{\eta}}^{n+1} \cdot \mathbf{w}^S \omega, \quad \forall \mathbf{w}^S = 0 \text{ on } \Gamma_D$$

$$\ddot{\mathbf{u}}^{n+1} = \dots$$

$$\mathbf{u}^{n+1} = \dots$$

Time integration shema for fluid equations

Find $\mathbf{v}^{n+1}$ and p^{n+1} such that

$$\begin{aligned} & \int_{\Omega_{t_{n+1}}^F} \rho^F \left(\frac{\mathbf{v}^{n+1} - \mathbf{V}^n}{\Delta t} \right) \cdot \mathbf{w}^F \\ & + \int_{\Omega_{t_{n+1}}^F} \rho^F \left(((\mathbf{V}^n - \mathbf{v}^{n+1}) \cdot \nabla) \mathbf{v}^{n+1} \right) \cdot \mathbf{w}^F \\ & \quad + a_F \left(\mathbf{v}^{n+1}, \mathbf{w}^F \right) + b_F \left(\mathbf{w}^F, p^{n+1} \right) \\ = & \int_{\Omega_{t_{n+1}}^F} \mathbf{f}^F \cdot \mathbf{w}^F + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \mathbf{w}^F + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot \mathbf{w}^F \\ & - \int_{\Gamma_{t_{n+1}}} \eta^{n+1} \cdot \mathbf{w}^F, \quad \forall \mathbf{w}^F \cdot \mathbf{n}^F = 0 \text{ on } \Sigma_2 \end{aligned}$$

$$b_F \left(\mathbf{v}^{n+1}, q \right) = 0, \quad \forall q$$

where $\mathbf{V}^n(\mathbf{x}) = \mathbf{v}^n \left(\mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n+1}}^{-1}(\mathbf{x}) \right)$

Concise form

Find $\mathbf{v}^{n+1}$ and p^{n+1} such that

$$\begin{aligned}\mathcal{A}_F(\mathbf{v}^{n+1}, \mathbf{w}^F) + b_F(\mathbf{w}^F, p^{n+1}) &= \mathcal{L}_F(\mathbf{w}^F) - \int_{\Gamma_{t_{n+1}}} \boldsymbol{\eta}^{n+1} \cdot \mathbf{w}^F, \\ &\quad \forall \mathbf{w}^F \cdot \mathbf{n}^F = 0 \text{ on } \Sigma_2 \\ b_F(\mathbf{v}^{n+1}, q) &= 0, \quad \forall q\end{aligned}$$

Augmented Lagrangian Method

Step 1. Solve fluid problem: find $\mathbf{v}^F$ and p^F such that

$$\begin{aligned}\mathcal{A}_F(\mathbf{v}^F, \mathbf{w}^F) + b_F(\mathbf{w}^F, p^F) &= \mathcal{L}_F(\mathbf{w}^F) - \int_{\Gamma_{\tilde{u}}} \boldsymbol{\lambda} \cdot \mathbf{w}^F \\ &\quad - r \int_{\Gamma_{\tilde{u}}} (\mathbf{v}^F - \mathbf{v}_{old}^S) \cdot \mathbf{w}^F, \\ b_F(\mathbf{v}^F, q) &= 0,\end{aligned}$$

Step 2. Solve structure problem: find $\hat{\mathbf{v}}^S$ such that

$$\mathcal{A}_S(\hat{\mathbf{v}}^S, \mathbf{w}^S) = \mathcal{L}_S(\mathbf{w}^S) + \int_{\Gamma_0} \hat{\boldsymbol{\lambda}} \cdot \mathbf{w}^S \omega + r \int_{\Gamma_0} (\hat{\mathbf{v}}^F - \hat{\mathbf{v}}^S) \cdot \mathbf{w}^S \omega$$

Step 3. If $\|\hat{\mathbf{v}}^F - \hat{\mathbf{v}}^S\|_{\Gamma_0} < tol$ then break

else $\hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{\lambda}} + \rho(\hat{\mathbf{v}}^F - \hat{\mathbf{v}}^S)$; $\mathbf{v}_{old}^S = \hat{\mathbf{v}}^S$; goto Step 1.

Semi-implicit time advancing scheme. From n to $n + 1$

Step 1. Explicit prediction $\tilde{\mathbf{u}}^{n+1} = \mathbf{u}^n + \Delta t \dot{\mathbf{u}}^n + \frac{(\Delta t)^2}{2} \ddot{\mathbf{u}}^n$

Step 2. Harmonic extension $\Delta \tilde{\mathbf{d}}^{n+1} = 0$, $\tilde{\mathbf{d}}^{n+1} = \tilde{\mathbf{u}}^{n+1}$ on Γ_0 ,
 $\tilde{\mathbf{d}}^{n+1} = 0$ on $\Sigma_1 \cup \Sigma_2 \cup \Sigma_3$

Step 3. Build mesh $\tilde{\mathcal{T}}_h^{n+1} = \mathbb{T}(\hat{\mathcal{T}}_h)$, where $\mathbb{T}(\hat{\mathbf{x}}) = \hat{\mathbf{x}} + \tilde{\mathbf{d}}^{n+1}(\hat{\mathbf{x}})$

Step 4. Mesh velocity $\tilde{\vartheta}^{n+1}(\mathbf{x}) = \frac{\tilde{\mathbf{d}}^{n+1}(\hat{\mathbf{x}}) - \tilde{\mathbf{d}}^n(\hat{\mathbf{x}})}{\Delta t}$

Step 5. $\mathbf{v}_{old}^S = \dot{\mathbf{u}}^n$, $\hat{\lambda} = \hat{\eta}^n$

Step 7. Solve fluid-structure problem by the Augmented Lagrangian Method in the fixed mesh $\tilde{\mathcal{T}}_h^{n+1}$

Step 8. Update

$$\mathbf{v}^{n+1} = \mathbf{v}^F, \quad p^{n+1} = p^F$$

$$\hat{\eta}^{n+1} = \hat{\lambda}$$

$$\dot{\mathbf{u}}^{n+1} = \hat{\mathbf{v}}^S, \quad \ddot{\mathbf{u}}^{n+1} = \dots, \quad \mathbf{u}^{n+1} = \dots$$

Physical and numerical parameters

The length is $L = 6 \text{ cm}$.

The inflow Σ_1 and outflow Σ_3 sections are segments of length 1.

The thickness of wall $h = 0.1 \text{ cm}$, the Young's modulus $E = 0.75 \cdot 10^6 \frac{\text{g}}{\text{cm} \cdot \text{s}^2}$, the Poisson's ratio $\nu = 0.3$ and the mass density $\rho^S = 1.1 \frac{\text{g}}{\text{cm}^3}$.

The viscosity of the fluid $\mu = 0.035 \frac{\text{g}}{\text{cm} \cdot \text{s}}$, the volume force in fluid is $\mathbf{f}^F = (0, 0)^T$, the outflow traction $\mathbf{h}_{out} = (0, 0)^T$ and the inflow traction $\mathbf{h}_{in}(t) = (2000 (1 - \cos(\frac{2\pi t}{0.025})), 0)^T$ if $0 \leq t \leq 0.025$ and $\mathbf{h}_{in}(t) = (0, 0)^T$ if $t > 0.025$.

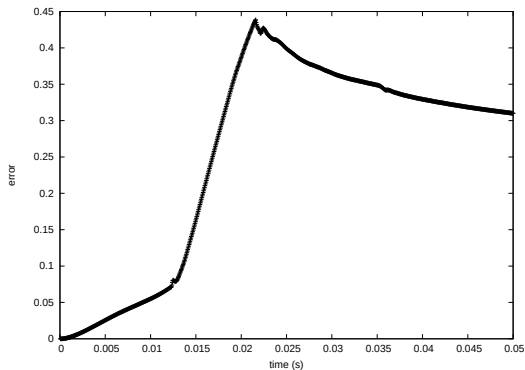
Newmark parameters: $\delta = 0.5$, $\theta = 0.25$

$\Delta t = 5 \cdot 10^{-5}$, $N = 1000$, $T = 0.05$

Penalization parameters $r = \rho = 10^5$

Triangular finite elements: *fluid* velocity $\mathbb{P}_1 + \text{bubble}$ and pressure $\mathbb{P}_1$, *structure*: $\mathbb{P}_1$

Computed $\|\hat{\mathbf{v}}^F - \hat{\mathbf{v}}^S\|_{\Gamma_0}$



No of iterations at one time step: 20, but the fluid and structure matrices do not change.

Concluding remarks

- ▶ Semi-implicit time advancing scheme is attractive.
- ▶ The equality of the velocities on the interface was treated by the Augmented Lagrangian Method.
- ▶ Investigate the dissipation source.
- ▶ Study the stability of the algorithm.