Numerical simulation of blood-artery interaction

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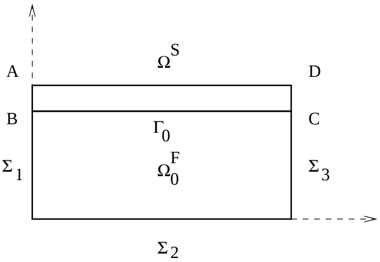
Initial and intermediate geometrical configurations



$$\partial\Omega_0^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_0, \qquad \partial\Omega_t^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_t.$$

Linear elasticity equations

Find the displacement $\mathbf{u} = (u_1, u_2)^T : \Omega^S \times [0, T] \to \mathbb{R}^2$ of the structure such that

$$
\begin{aligned}
\rho^S \frac{\partial^2 \mathbf{u}}{\partial t^2} - \nabla \cdot \sigma^S &= \mathbf{f}^S, \quad \text{in } \Omega^S \times (0, T) \\
\sigma^S &= \lambda^S (\nabla \cdot \mathbf{u}) \mathbb{I}_2 + 2\mu^S \epsilon(\mathbf{u}) \\
\epsilon(\mathbf{u}) &= \frac{1}{2}\left(\nabla \mathbf{u} + (\nabla \mathbf{u})^T\right) \\
\mathbf{u} &= 0, \quad \text{on } \Gamma_D \times (0, T) \\
\sigma^S \mathbf{n}^S &= 0, \quad \text{on } \Gamma_N \times (0, T)
\end{aligned}
$$

$$\Gamma_D = [AB] \cup [CD], \quad \Gamma_N = [DA].$$

Navier-Stokes equations

Find the velocity $\mathbf{v}$ and the pressure p of the fluid such that

$$
\begin{aligned}
\rho^F\left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) - \nabla\cdot\sigma^F &= \mathbf{f}^F, \quad \forall t\in(0,T), \forall \mathbf{x}\in\Omega_t^F \\
\nabla\cdot\mathbf{v} &= 0, \quad \forall t\in(0,T), \forall \mathbf{x}\in\Omega_t^F \\
\sigma^F &= -p\mathbb{I}_2 + 2\mu^F\epsilon(\mathbf{v}) \\
\sigma^F\mathbf{n}^F &= \mathbf{h}_{in}, \quad \text{on } \Sigma_1\times(0,T) \\
\sigma^F\mathbf{n}^F &= \mathbf{h}_{out}, \quad \text{on } \Sigma_3\times(0,T) \\
\mathbf{v}\cdot\mathbf{n}^F &= 0, \quad \text{on } \Sigma_2\times(0,T) \\
\boldsymbol{\tau}^F\cdot\left(\sigma^F\mathbf{n}^F\right) &= 0, \quad \text{on } \Sigma_2\times(0,T)
\end{aligned}
$$

Interface and initial conditions

The interface Γ_t is the image of the boundary Γ_0 by the map

$$\mathbb{T}(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X}, t).$$

Interface conditions

$$\begin{aligned}
\mathbf{v}(\mathbf{X} + \mathbf{u}(\mathbf{X}, t), t) &= \frac{\partial \mathbf{u}}{\partial t}(\mathbf{X}, t), \ \forall (\mathbf{X}, t) \in \Gamma_0 \times (0, T) \\
\left(\sigma^F \mathbf{n}^F\right)_{(\mathbf{X}+\mathbf{u}(\mathbf{X},t),t)} \omega(\mathbf{X}, t) &= -\left(\sigma^S \mathbf{n}^S\right)_{(\mathbf{X},t)}, \ \forall (\mathbf{X}, t) \in \Gamma_0 \times (0, T)
\end{aligned}$$

where $\omega(\mathbf{X}, t) = \left\|\mathrm{cof}(\nabla \mathbb{T}) \mathbf{n}^S\right\|_{\mathbb{R}^2}$.

Initial conditions

$$\begin{aligned}
\mathbf{u}(\mathbf{X}, t = 0) &= \mathbf{u}^0(\mathbf{X}), \ \text{in } \Omega^S \\
\frac{\partial \mathbf{u}}{\partial t}(\mathbf{X}, t = 0) &= \dot{\mathbf{u}}^0(\mathbf{X}), \ \text{in } \Omega^S \\
\mathbf{v}(\mathbf{x}, t = 0) &= \mathbf{v}^0(\mathbf{x}), \ \text{in } \Omega_0^F
\end{aligned}$$

Difficulties

- flow in a moving domain
- coupled equations
- Lagrangian coordinates for the structure (artery) and Eulerian coordinates for the fluid (blood)
- strongly non-linear system
- pulsating flow

Approximation

- artery: Finite Element and modal decomposition
- blood: Arbitrary Lagrangian Eulerian framework and Finite Element
- quasi Newton method in order to get at each time step, the continuity of the velocity and of the stress at the interface
- partitioned procedures
- not matching meshes

Time advancing schemes

- implicit:
  Ω^F_{n+1}, $\mathbf{v}^{n+1}$, p^{n+1}, $\mathbf{u}^{n+1}$ have to be computed simultaneous
- semi-implicit:
  - explicit prediction of the fluid domain
    $\Omega^F_{n+1} = (\mathbb{I} + \Delta t \boldsymbol{\vartheta}^n)\left(\Omega^F_n\right)$
  - after that, $\mathbf{v}^{n+1}$, p^{n+1}, $\mathbf{u}^{n+1}$ have to be computed simultaneous

Physical and numerical parameters

The length is $L = 6\ cm$. The inflow Σ_1 and outflow Σ_3 sections are segments of length 1.

The thickness of wall $h = 0.1\ cm$, the Young's modulus $E = 3 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the Poisson's ratio $\nu = 0.3$ and the mass density $\rho^S = 1.1\ \frac{g}{cm^3}$.

The viscosity of the fluid $\mu = 0.035\ \frac{g}{cm \cdot s}$, the volume force in fluid is $\mathbf{f}^F = (0,0)^T$, the outflow traction $\mathbf{h}_{out} = (0,0)^T$ and the inflow traction $\mathbf{h}_{in}(t) = \left(1000\left(1 - cos(\frac{2\pi t}{0.025})\right), 0\right)^T$ if $0 \leq t \leq 0.025$ and $\mathbf{h}_{in}(t) = (0,0)^T$ if $t > 0.025$.

$\Delta t = 10^{-3}$, $N = 100$, $T = 0.1$