Monolithic algorithm for dynamic fluid-structure interaction problem

Cornel Marius Murea *

Abstract

We propose a numerical method for a fluid-structure interaction problem. Updated Lagrangian method is used for the structure and fluid equations are written in Arbitrary Lagrangian Eulerian coordinates. The global moving mesh for the fluid-structure domain is aligned with the fluid-structure interface. At each time step, we solve a monolithic system of unknowns velocity and pressure defined on the global mesh. The continuity of velocity at the interface is automatically satisfied, while the continuity of stress does not appear explicitly in the monolithic fluid-structure system. At each time step we solve only one linear system. Numerical results are presented.

1 Setting the fluid-structure interaction problem

We study a two dimensional fluid-structure interaction problem. We denote by Ω_0^S the initial structure domain and we assume that its boundary admits the decomposition $\partial\Omega_0^S = \Gamma_D \cup \Gamma_0$. We suppose that the initial structure domain is undeformed (stress-free). At the time instant t, the structure occupies the domain Ω_t^S bounded by $\partial\Omega_t^S = \Gamma_D \cup \Gamma_t$. On the boundary Γ_D, we impose zero displacements.

Let D be a rectangle of boundary $\partial D = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Sigma_4$, with Σ_1 the left, Σ_2 the bottom, Σ_3 the right and Σ_4 the top boundary, (see Figure 1).

We assume that the structure is completely embedded into the fluid, therefore at the time instant t, the fluid occupies the domain $\Omega_t^F = D \setminus \overline{\Omega}_t^S$. The boundary $\partial\Omega_t^S$ is common of both domains.

We denote by $\mathbf{U}^S : \Omega_0^S \times [0,T] \to \mathbb{R}^2$ the displacement of the structure. A particle of the structure whose initial position was the point $\mathbf{X}$ will occupies the position $\mathbf{x} = \mathbf{X} + \mathbf{U}^S(\mathbf{X},t)$ in the deformed domain Ω_t^S.

*Laboratoire de Mathématiques, Informatique et Applications, Université de Haute Alsace, 6, rue des Frères Lumière, 68093 Mulhouse Cedex, France, e-mail: `cornel.murea@uha.fr`

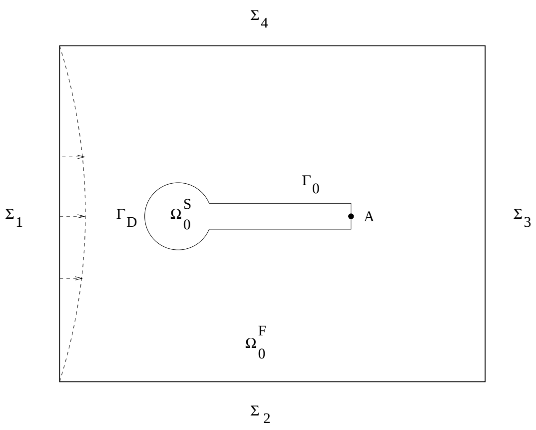

Figure 1: Geometrical configuration

We denote by $\mathbf{F}(\mathbf{X},t) = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{U}^S(\mathbf{X},t)$ the gradient of the deformation, where $\mathbf{I}$ is the unity matrix and we set $J(\mathbf{X},t) = \det \mathbf{F}(\mathbf{X},t)$.

The first and the second Piola-Kirchhoff stress tensors are denoted by $\mathbf{\Pi}$ and $\mathbf{\Sigma}$, respectively and the following equality holds $\mathbf{\Pi} = \mathbf{F\Sigma}$. We suppose that the material of the structure is elastic, homogeneous, isotropic.

We have assumed that the fluid is governed by the Navier-Stokes equations. For each time instant $t \in [0,T]$, we denote the fluid velocity by $\mathbf{v}^F(t) = \left(v_1^F(t), v_2^F(t)\right)^T : \Omega_t^F \to \mathbb{R}^2$ and the fluid pressure by $p^F(t) : \Omega_t^F \to \mathbb{R}$. Let us remark that the fluid domain Ω_t^F depends on the position of the interface Γ_t, which is the image of Γ_0 via the map $\mathbf{X} \to \mathbf{X} + \mathbf{U}^S(\mathbf{X},t)$.

Let $\epsilon\left(\mathbf{v}^F\right) = \frac{1}{2}\left(\nabla\mathbf{v}^F + \left(\nabla\mathbf{v}^F\right)^T\right)$ be the fluid rate of strain tensor and let $\sigma^F = -p^F\mathbf{I} + 2\mu^S\epsilon\left(\mathbf{v}^F\right)$ be the fluid stress tensor. In order to simplify the notation, we write $\nabla\mathbf{v}^F$ in place of $\nabla_{\mathbf{x}}\mathbf{v}^F$, when the gradients are computed with respect to the Eulerian coordinates $\mathbf{x}$.

The problem is to find the structure displacement $\mathbf{U}^S$, the fluid velocity $\mathbf{v}^F$ and the fluid pressure p^F such that:

$$\rho_0^S(\mathbf{X})\frac{\partial^2\mathbf{U}^S}{\partial t^2}(\mathbf{X},t) - \nabla_{\mathbf{X}}\cdot(\mathbf{F\Sigma})(\mathbf{X},t) = \rho_0^S(\mathbf{X})\,\mathbf{g}, \quad \text{in } \Omega_0^S \times (0,T) \tag{1}$$

$$\mathbf{U}^S(\mathbf{X},t) = 0, \quad \text{on } \Gamma_D \times (0,T) \tag{2}$$

$$\rho^F\left(\frac{\partial \mathbf{v}^F}{\partial t}+(\mathbf{v}^F\cdot\nabla)\mathbf{v}^F\right)-2\mu^F\nabla\cdot\epsilon\left(\mathbf{v}^F\right)+\nabla p^F = \rho^F\mathbf{g},\ \forall t\in(0,T),\forall\mathbf{x}\in\Omega_t^F \quad (3)$$

$$\nabla\cdot\mathbf{v}^F = 0,\ \forall t\in(0,T),\forall\mathbf{x}\in\Omega_t^F \quad (4)$$

$$\mathbf{v} = \mathbf{v}_{in},\ \text{on } \Sigma_1\times(0,T) \quad (5)$$

$$\sigma^F\mathbf{n}^F = \mathbf{h}_{out},\ \text{on } \Sigma_3\times(0,T) \quad (6)$$

$$\mathbf{v}^F = 0,\ \text{on } \Sigma_2\cup\Sigma_4\cup\Gamma_D \quad (7)$$

$$\mathbf{v}^F\left(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t\right) = \frac{\partial\mathbf{U}^S}{\partial t}(\mathbf{X},t),\ \text{on } \Gamma_0\times(0,T) \quad (8)$$

$$\left(\sigma^F\mathbf{n}^F\right)_{(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t)} = -(\mathbf{F\Sigma})(\mathbf{X},t)\,\mathbf{N}^S(\mathbf{X}),\text{on } \Gamma_0\times(0,T) \quad (9)$$

$$\mathbf{U}^S(\mathbf{X},0) = \mathbf{U}^{S,0}(\mathbf{X}),\ \text{in } \Omega_0^S \quad (10)$$

$$\frac{\partial\mathbf{U}^S}{\partial t}(\mathbf{X},0) = \mathbf{V}^{S,0}(\mathbf{X}),\ \text{in } \Omega_0^S \quad (11)$$

$$\mathbf{v}^F(\mathbf{X},0) = \mathbf{v}^{F,0}(\mathbf{X}),\ \text{in } \Omega_0^F \quad (12)$$

where $\rho_0^S:\Omega_0^S\to\mathbb{R}$ is the initial mass density of the structure, $\mathbf{g}$ is the acceleration of gravity vector and it is assumed to be constant, $\mathbf{N}^S$ is the unit outer normal vector along the boundary $\partial\Omega_0^S$, $\rho^F>0$ and $\mu^F>0$ are constants and its represent the mass density and the viscosity of the fluid, $\mathbf{v}_{in}$ is the prescribed inflow velocity, $\mathbf{h}_{out}$ is prescribed outflow boundary stress, $\mathbf{n}^F$ is the unit outer normal vector along the boundary $\partial\Omega_t^F$.

For the structure equations (1)–(2), we have used the Lagrangian coordinates, while for the fluid equations (3)–(7) the Eulerian coordinates have been used. The equations (8) and (9) represent the continuity of velocity and of stress at the interface, respectively. Initial conditions are given by (10)–(12). The governing equations for fluid-structure interaction are (1)–(12).

2 Total Lagrangian framework for the structure approximation

Let us introduce $\mathbf{V}^S$ the velocity of the structure in the Lagrangian coordinates. The equation (1) is equivalent to

$$\rho_0^S(\mathbf{X})\frac{\partial\mathbf{V}^S}{\partial t}(\mathbf{X},t)-\nabla_{\mathbf{X}}\cdot(\mathbf{F\Sigma})(\mathbf{X},t) = \rho_0^S(\mathbf{X})\,\mathbf{g},\quad \text{in } \Omega_0^S\times(0,T) \quad (13)$$

$$\frac{\partial\mathbf{U}^S}{\partial t}(\mathbf{X},t) = \mathbf{V}^S(\mathbf{X},t),\quad \text{in } \Omega_0^S\times(0,T). \quad (14)$$

Let $N \in \mathbb{N}^*$ be the number of time steps and $\Delta t = T/N$ the time step. We set $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$. Let $\mathbf{V}^{S,n}(\mathbf{X})$ and $\mathbf{U}^{S,n}(\mathbf{X})$ be approximations of $\mathbf{V}^{S}(\mathbf{X}, t_n)$ and $\mathbf{U}^{S}(\mathbf{X}, t_n)$. We also use following notations

$$\mathbf{F}^n = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^{S,n}, \quad \mathbf{\Sigma}^n = \mathbf{\Sigma}(\mathbf{F}^n),\ n \geq 0.$$

The system (13)–(14) will be approached by the implicit Euler scheme

$$\rho_0^S(\mathbf{X}) \frac{\mathbf{V}^{S,n+1}(\mathbf{X}) - \mathbf{V}^{S,n}(\mathbf{X})}{\Delta t} - \nabla_{\mathbf{X}} \cdot \left(\mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}\right)(\mathbf{X}) = \rho_0^S(\mathbf{X})\,\mathbf{g}, \text{ in } \Omega_0^S \quad (15)$$

$$\frac{\mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X})}{\Delta t} = \mathbf{V}^{S,n+1}(\mathbf{X}), \text{ in } \Omega_0^S \quad (16)$$

From (16), we get $\mathbf{F}^{n+1} = \mathbf{F}^n + \Delta t \nabla_{\mathbf{X}} \mathbf{V}^{S,n+1}$ and consequently, $\mathbf{F}^{n+1}$ and $\mathbf{\Sigma}^{n+1}$ depend on the velocity $\mathbf{V}^{S,n+1}$ but not in the displacement $\mathbf{U}^{S,n+1}$. In other word, we have eliminated the unknown displacement and we have now an equation of unknown $\mathbf{V}^{S,n+1}$.

The weak form of the equation (15) is: find $\mathbf{V}^{S,n+1} : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{V}^{S,n+1} = 0$ on Γ_D, such that

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S \, d\mathbf{X} + \int_{\Omega_0^S} \mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}} \mathbf{W}^S \, d\mathbf{X}$$
$$= \int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S \, d\mathbf{X} + \int_{\Gamma_0} \mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}\mathbf{N}^S \cdot \mathbf{W}^S \, dS \quad (17)$$

for all $\mathbf{W}^S : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{W}^S = 0$ on Γ_D. For instant, we have assumed that the forces $\mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}\mathbf{N}^S$ on the interface Γ_0 are known.

3 Updated Lagrangian framework for the structure approximation

We follow a similar approach that in [3], where the structure is a Neo-Hookean material. In the present paper, the structure is governed by the linear elasticity equations. We denote by Ω_n^S the image of Ω_0^S via the map $\mathbf{X} \to \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X})$ and we set $\widehat{\Omega}^S = \Omega_n^S$ the computational domain for the structure.

The map from Ω_0^S to Ω_{n+1}^S defined by $\mathbf{X} \to \mathbf{x} = \mathbf{X} + \mathbf{U}^{S,n+1}(\mathbf{X})$ is the composition of the map from Ω_0^S to $\widehat{\Omega}^S$ defined by $\mathbf{X} \to \widehat{\mathbf{x}} = \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X})$ with the map from $\widehat{\Omega}^S$ to Ω_{n+1}^S defined by

$$\widehat{\mathbf{x}} \to \mathbf{x} = \widehat{\mathbf{x}} + \mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X}) = \widehat{\mathbf{x}} + \widehat{\mathbf{u}}(\widehat{\mathbf{x}}).$$

With the notations $\widehat{\mathbf{F}} = \mathbf{I} + \nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{u}}$ and $\widehat{J} = \det \widehat{\mathbf{F}}$, $J^n = \det \mathbf{F}^n$, we obtain

$$\mathbf{F}^{n+1}(\mathbf{X}) = \widehat{\mathbf{F}}(\widehat{\mathbf{x}})\,\mathbf{F}^n(\mathbf{X}), \quad J^{n+1}(\mathbf{X}) = \widehat{J}(\widehat{\mathbf{x}})\,J^n(\mathbf{X}). \tag{18}$$

The relation between the Cauchy stress tensor of the structure σ^S and the second Piola-Kirchhoff stress tensor $\mathbf{\Sigma}$ is the following $\sigma^S(\mathbf{x},t) = \left(\frac{1}{J}\mathbf{F}\mathbf{\Sigma}\mathbf{F}^T\right)(\mathbf{X},t)$, where $\mathbf{x} = \mathbf{X} + \mathbf{U}^S(\mathbf{X},t)$. The mass conservation assumption gives $\rho^S(\mathbf{x},t) = \frac{\rho_0^S(\mathbf{X})}{J(\mathbf{X},t)}$, where $\rho^S(\mathbf{x},t)$ is the mass density of the structure in the Eulerian framework.

For the semi-discrete scheme, we use the notations

$$\sigma^{S,n+1}(\mathbf{x}) = \left(\frac{1}{J^{n+1}}\mathbf{F}^{\mathbf{n+1}}\mathbf{\Sigma}^{n+1}\left(\mathbf{F}^{n+1}\right)^T\right)(\mathbf{X}), \quad \mathbf{x} = \mathbf{X} + \mathbf{U}^{S,n+1}(\mathbf{X})$$

and $\rho^{S,n}(\widehat{\mathbf{x}}) = \frac{\rho_0^S(\mathbf{X})}{J^n(\mathbf{X})}, \quad \widehat{\mathbf{x}} = \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X})$.

Let us introduce $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$ defined by $\widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}}) = \mathbf{V}^{S,n+1}(\mathbf{X})$ and $\mathbf{v}^{S,n}(\widehat{\mathbf{x}}) = \mathbf{V}^{S,n}(\mathbf{X})$. Also, for $\mathbf{W}^S : \Omega_0^S \to \mathbb{R}^2$, we define $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2$ and $\mathbf{w}^S : \Omega_{n+1}^S \to \mathbb{R}^2$ by $\widehat{\mathbf{w}}^S(\widehat{\mathbf{x}}) = \mathbf{w}^S(\mathbf{x}) = \mathbf{W}^S(\mathbf{X})$.

Now, we rewrite the equation (17) over the domain $\widehat{\Omega}^S$. For the first term of (17), we get

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S\, d\mathbf{X} = \int_{\widehat{\Omega}^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S\, d\widehat{\mathbf{x}}$$

and similarly

$$\int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S\, d\mathbf{X} = \int_{\widehat{\Omega}^S} \rho^{S,n}\mathbf{g} \cdot \widehat{\mathbf{w}}^S\, d\widehat{\mathbf{x}}.$$

Using the identity $\left(\nabla \mathbf{w}^S(\mathbf{x})\right)\mathbf{F}^{n+1}(\mathbf{X}) = \nabla_{\mathbf{X}}\mathbf{W}^S(\mathbf{X})$ and the definition of $\sigma^{S,n+1}$, we get

$$\int_{\Omega_0^S} \mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}}\mathbf{W}^S\, d\mathbf{X} = \int_{\Omega_{n+1}^S} \sigma^{S,n+1} : \nabla \mathbf{w}^S\, d\mathbf{x}.$$

Details about this kind of transformation could be found in [1], Chapter 1.2.

In order to write the above integral over the domain $\widehat{\Omega}^S$, let us introduce the tensor

$$\widehat{\mathbf{\Sigma}}(\widehat{\mathbf{x}}) = \widehat{J}(\widehat{\mathbf{x}})\,\widehat{\mathbf{F}}^{-1}(\widehat{\mathbf{x}})\,\sigma^{S,n+1}(\mathbf{x})\,\widehat{\mathbf{F}}^{-T}(\widehat{\mathbf{x}}). \tag{19}$$

Since $\left(\nabla \mathbf{w}^S(\mathbf{x})\right)\widehat{\mathbf{F}}(\widehat{\mathbf{x}}) = \nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}^S(\widehat{\mathbf{x}})$, see [1], Chapter 1.2 and taking into account (19), we get

$$\int_{\Omega_{n+1}^S} \sigma^{S,n+1} : \nabla \mathbf{w}^S\, d\mathbf{x} = \int_{\widehat{\Omega}^S} \widehat{\mathbf{F}}\widehat{\mathbf{\Sigma}} : \nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}^S\, d\widehat{\mathbf{x}}.$$

Now, it is possible to present the updated Lagrangian version of (17). Knowing $\mathbf{U}^{S,n} : \Omega_0^S \to \mathbb{R}^2$, $\widehat{\Omega}^S = \Omega_n^S$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$, we try to find $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$,

$\widehat{\mathbf{v}}^{S,n+1} = 0$ on Γ_D such that

$$\int_{\widehat{\Omega}^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$
$$= \int_{\widehat{\Omega}^S} \rho^{S,n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\Gamma_0} \mathbf{F}^{n+1} \boldsymbol{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS \tag{20}$$

for all $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{w}}^S = 0$ on Γ_D. We recall that the forces $\mathbf{F}^{n+1}\boldsymbol{\Sigma}^{n+1}\mathbf{N}^S$ on the interface Γ_0 are assumed known.

Using the identity $\widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = \mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X}) = \Delta t \, \mathbf{V}^{S,n+1}(\mathbf{X}) = \Delta t \, \widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}})$, we obtain

$$\widehat{\mathbf{F}} = \mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}. \tag{21}$$

Using (18) and (19), it follows that

$$\begin{aligned} \widehat{\boldsymbol{\Sigma}} &= \widehat{J}\widehat{\mathbf{F}}^{-1}\sigma^{S,n+1}\widehat{\mathbf{F}}^{-T} = \widehat{J}\widehat{\mathbf{F}}^{-1}\frac{1}{J^{n+1}}\mathbf{F}^{n+1}\boldsymbol{\Sigma}^{n+1}\left(\mathbf{F}^{n+1}\right)^T\widehat{\mathbf{F}}^{-T} \\ &= \frac{1}{J^n}\mathbf{F}^n\boldsymbol{\Sigma}^{n+1}\left(\mathbf{F}^n\right)^T. \end{aligned} \tag{22}$$

For the linear elastic material, we have $\boldsymbol{\Sigma}(\mathbf{U}) = \lambda^S(\nabla_{\mathbf{X}} \cdot \mathbf{U}) + \mu^S\left(\nabla_{\mathbf{X}}\mathbf{U} + (\nabla_{\mathbf{X}}\mathbf{U})^T\right)$ where λ^S and μ^S are the Lamé coefficients. We have $\boldsymbol{\Sigma}^{n+1} = \boldsymbol{\Sigma}(\mathbf{U}^{S,n+1}) = \boldsymbol{\Sigma}(\mathbf{U}^{S,n}) + (\Delta t)\boldsymbol{\Sigma}(\mathbf{V}^{S,n+1}) = \boldsymbol{\Sigma}^n + (\Delta t)\boldsymbol{\Sigma}(\mathbf{V}^{S,n+1})$.

We introduce $\boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{u}}) = \lambda^S(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{u}}) + \mu^S\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{u}} + (\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{u}})^T\right)$ and $\boldsymbol{\Sigma}(\mathbf{V}^{S,n+1})$ could be approached by $\boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{S,n+1})$. We can approach the map $\widehat{\mathbf{v}}^{S,n+1} \to \widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}}$ by the linear application

$$\begin{aligned} \widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} &\approx \frac{1}{J^n}\mathbf{F}^n\boldsymbol{\Sigma}^n\left(\mathbf{F}^n\right)^T + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\frac{1}{J^n}\mathbf{F}^n\boldsymbol{\Sigma}^n\left(\mathbf{F}^n\right)^T + \frac{\Delta t}{J^n}\mathbf{F}^n\boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{S,n+1})\left(\mathbf{F}^n\right)^T \\ &= \sigma^{S,n} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\sigma^{S,n} + \frac{\Delta t}{J^n}\mathbf{F}^n\boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{S,n+1})\left(\mathbf{F}^n\right)^T. \end{aligned}$$

We define $\widehat{\mathbf{u}}^{S,n}(\widehat{\mathbf{x}}) = \mathbf{U}^{S,n}(\mathbf{X})$ and for the small deformations, we have $\mathbf{F}^n \approx \mathbf{I}$, $J^n \approx 1$, $\sigma^{S,n} \approx \boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{u}}^{S,n+1})$. Finally, we replace the map $\widehat{\mathbf{v}}^{S,n+1} \to \widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}}$ by the linear application

$$\widehat{\mathbf{L}}\left(\widehat{\mathbf{v}}^{S,n+1}\right) = \boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{u}}^{S,n}) + (\Delta t)\boldsymbol{\Sigma}_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{S,n+1}). \tag{23}$$

The linearized updated Lagrangian weak formulation of the structure is: knowing $\mathbf{U}^{S,n} : \Omega_0^S \to \mathbb{R}^2$, $\widehat{\Omega}^S = \Omega_n^S$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$, find $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{S,n+1} = 0$ on Γ_D such that

$$\int_{\widehat{\Omega}^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \widehat{\mathbf{L}}\left(\widehat{\mathbf{v}}^{S,n+1}\right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$
$$= \int_{\widehat{\Omega}^S} \rho^{S,n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\Gamma_0} \mathbf{F}^{n+1} \boldsymbol{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS \tag{24}$$

for all $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{w}}^S = 0$ on Γ_D.

4 Monolithic algorithm for the fluid-structure equations

We have $\partial\Omega_n^S = \Gamma_D \cup \Gamma_n$, where Γ_n is a approximation of the moving interface Γ_{t_n}, $\Omega_n^F = D \setminus \overline{\Omega}_n^S$ and let us introduce the global velocity, pressure and test function

$$\widehat{\mathbf{v}}^{n+1} : D \to \mathbb{R}^2, \quad \widehat{p}^{n+1} : D \to \mathbb{R}, \quad \widehat{\mathbf{w}} : D \to \mathbb{R}^2$$

$$\widehat{\mathbf{v}}^{n+1} = \begin{cases} \widehat{\mathbf{v}}^{F,n+1} \text{ in } \Omega_n^F \\ \widehat{\mathbf{v}}^{S,n+1} \text{ in } \Omega_n^S \end{cases} \quad \widehat{p}^{n+1} = \begin{cases} \widehat{p}^{F,n+1} \text{ in } \Omega_n^F \\ \widehat{p}^{S,n+1} \text{ in } \Omega_n^S \end{cases} \quad \widehat{\mathbf{w}} = \begin{cases} \widehat{\mathbf{w}}^F \text{ in } \Omega_n^F \\ \widehat{\mathbf{w}}^S \text{ in } \Omega_n^S. \end{cases}$$

Algorithm for fluid-structure interaction
Time advancing scheme from n to $n+1$

We assume that we know the mesh $\mathcal{T}_h^n$, the velocity $\mathbf{v}^n$, the pressure p^n, and the mesh velocity $\boldsymbol{\vartheta}^n$.

Step 1: Solve the monolithic **linear** system and get the velocity $\widehat{\mathbf{v}}^{n+1} \in \left(H^1(D)\right)^2$, $\widehat{\mathbf{v}}^{n+1} = \mathbf{v}_{in}$ on Σ_1, $\widehat{\mathbf{v}}^{n+1} = 0$ on $\partial D \cup \Gamma_D$ and the pressure $\widehat{p}^{n+1} \in L^2(D)$, $\widehat{p}^{n+1} = 0$ in Ω_n^S, such that:

$$\begin{aligned}
&\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \left(\left(\left(\mathbf{v}^n - \boldsymbol{\vartheta}^n\right) \cdot \nabla_{\widehat{\mathbf{x}}}\right) \widehat{\mathbf{v}}^{n+1}\right) \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} \\
&- \int_{\Omega_n^F} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}\right) \widehat{p}^{n+1} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} 2\mu^F \epsilon\left(\widehat{\mathbf{v}}^{n+1}\right) : \epsilon\left(\widehat{\mathbf{w}}\right) d\widehat{\mathbf{x}} \\
&+ \int_{\Omega_n^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \widehat{\mathbf{L}}\left(\widehat{\mathbf{v}}^{n+1}\right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} \\
&= \int_{\Omega_n^F} \rho^F \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \mathbf{f}^{F,n} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Sigma_3} \mathbf{h}_{out} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} \\
&+ \int_{\Omega_n^S} \rho^{S,n} \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \rho^{S,n} \mathbf{g} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}}, \qquad (25)
\end{aligned}$$

$$\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{n+1}) \widehat{q}\, d\widehat{\mathbf{x}} = 0, \qquad (26)$$

for all $\widehat{\mathbf{w}} \in \left(H^1(D)\right)^2$, $\widehat{\mathbf{w}} = 0$ on $\partial D \cup \Gamma_D$ and for all $\widehat{q} \in L^2(D)$.

Step 2: Compute the mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1} : D \to \mathbb{R}^2$

$$\begin{cases} \Delta_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} = 0, & D \\ \widehat{\boldsymbol{\vartheta}}^{n+1} = 0, & \partial D \cup \Gamma_D \\ \widehat{\boldsymbol{\vartheta}}^{n+1} = \widehat{\mathbf{v}}^{n+1}, & \Gamma_n. \end{cases} \qquad (27)$$

We can replace in (27), the Laplacian by the linear elasticity operator in order to improve the quality of the mesh.

Step 3: Define the map $\mathbb{T}_n : \overline{D} \to \mathbb{R}^2$ by:

$$\mathbb{T}_n(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + (\Delta t)\widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}})\chi_{\Omega_n^F}(\widehat{\mathbf{x}}) + (\Delta t)\widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}})\chi_{\Omega_n^S}(\widehat{\mathbf{x}})$$

where $\chi_{\Omega_n^F}$ and $\chi_{\Omega_n^S}$ are the characteristic functions of fluid and structure domains.

The new mesh is $\mathbb{T}_n(\mathcal{T}_h^n) = \mathcal{T}_h^{n+1}$.

Step 4: We define $\mathbf{v}^{n+1} : D \to \mathbb{R}^2$, $p^{n+1} : D \to \mathbb{R}$ and $\boldsymbol{\vartheta}^{n+1} : D \to \mathbb{R}^2$ by:

$$\mathbf{v}^{n+1}(\mathbf{x}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}),\ p^{n+1}(\mathbf{x}) = \widehat{p}^{n+1}(\widehat{\mathbf{x}}),\ \boldsymbol{\vartheta}^{n+1}(\mathbf{x}) = \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}}),\ \forall \widehat{\mathbf{x}} \in D \text{ and } \mathbf{x} = \mathbb{T}_n(\widehat{\mathbf{x}}).$$

We solve the monolithic system (25)-(26) using globally continuous finite element for the velocity $\widehat{\mathbf{v}}^{n+1} \in \left(H^1(D)\right)^2$ defined all over the fluid-structure global mesh. Then the both continuity conditions at the interface hold. For the global pressure $\widehat{p}^{n+1} \in L^2(D)$, we have to impose $\widehat{p}^{n+1} = 0$ in Ω_n^S, more precisely we impose $\widehat{p}^{n+1} = 0$ at each node of the structure sub-domain excepting the nodes on the interface Γ_n.

This algorithm is similar to [4], where the Newmark method was employed for the structure, but the actual algorithm is not a particular case of the cited paper. In addition, the quality of the mesh is augmented in the actual version by solving the mesh velocity after the resolution of the monolithic linear system. Another improvement is that we use now the facility of `FreeFem++` to integrate over a sub-domain, which is faster that using the characteristic function.

5 Numerical test. Flow around a flexible thin structure attached to a fixed cylinder

We have tested the benchmark FSI3 from [5]. The numerical tests have been produced using *FreeFem++* (see [2]).

The structure is composed by a rectangular flexible beam attached to a fixed circle, see Figure 1. The circle center is positioned at $(0.2, 0.2)\ m$ measured from the left bottom corner of the channel. The circle has the radius $r = 0.5\ m$ and the rectangular beam is of length $\ell = 0.35\ m$, thickness $h = 0.02\ m$. The mass density is $\rho^S = 1000 Kg/m^3$, the Young modulus is $E^S = 5.6 \times 10^6\ Pa$ and the Poisson's ratio is $\nu^S = 0.4$.

The channel has the length $L = 2.5\ m$ and the width $H = 0.41\ m$. The fluid dynamic viscosity is $\mu^F = 1\ Kg/(m\,s)$ and the mass density is $\rho^F = 1000 Kg/m^3$.

We denote by $\Sigma_1 = \{0\} \times [0, H]$, $\Sigma_3 = \{L\} \times [0, H]$ the left and the right vertical boundaries of the channel and by $\Sigma_2 = [0, L] \times \{0\}$, $\Sigma_4 = [0, L] \times \{H\}$ the bottom and the top boundaries, respectively.

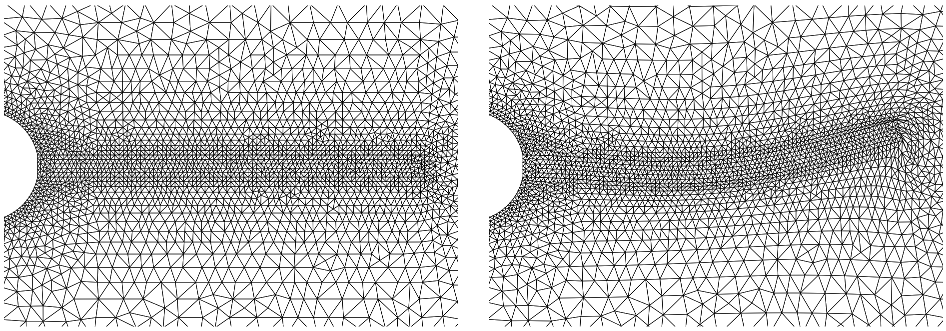

Figure 2: Details of the fluid-structure mesh at $t = 0$ and $t = 6.016$.

We have used the boundary condition $\mathbf{v} = \mathbf{v}_{in}$ at the inflow Σ_1, where

$$\mathbf{v}_{in}(x_1, x_2, t) = \begin{cases} \left(1.5\,\overline{U}\frac{x_2(H-x_2)}{(H/2)^2}\frac{(1-\cos(\pi t/2))}{2},\ 0\right), & (x_1, x_2) \in \Sigma_1, 0 \le t \le 2 \\ \left(1.5\,\overline{U}\frac{x_2(H-x_2)}{(H/2)^2},\ 0\right), & (x_1, x_2) \in \Sigma_1, 2 \le t \le T = 8 \end{cases}$$

and $\overline{U} = 2$. At Σ_2, Σ_4, as well as on the boundary of the circle, we have imposed the no-slip boundary condition $\mathbf{v} = \mathbf{0}$. At the outflow Σ_3, we have imposed the traction free $\sigma^F(\mathbf{v}, p)\,\mathbf{n}^F = 0$. Initially, the fluid and the structure are at rest.

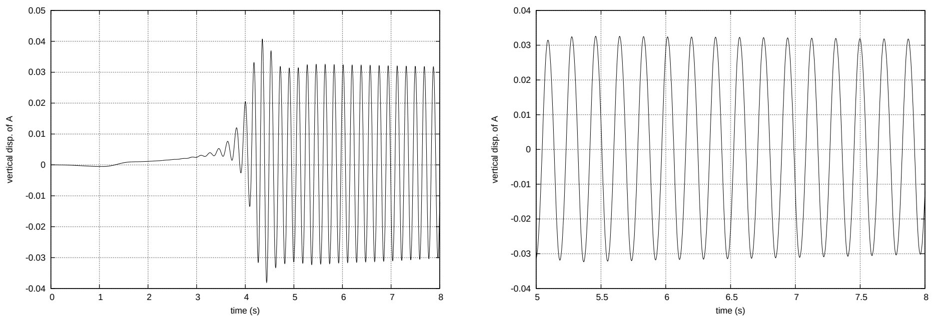

Figure 3: Time history of the vertical displacement of the point A.

We use a global mesh for the fluid-structure domain of 9382 triangles and 4859 vertices, see Figure 2. The time step is $\Delta t = 0.002\ s$ and the number of time steps is $N = 4000$. Using *FreeFem++* [2], it is possible to construct a global fluid-structure mesh with an "interior boundary" which is the fluid-structure interface. The global moving mesh for the fluid-structure domain is aligned with the fluid-structure interface and changes at at each time step. For the finite element approximation of the fluid-structure velocity, we have used the triangular finite element $\mathbb{P}_1 + bubble$ and we have

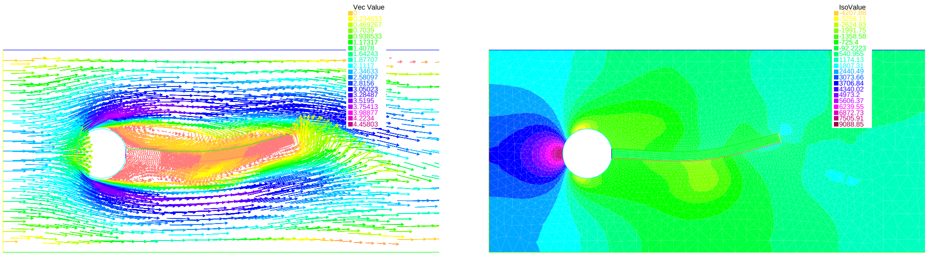

Figure 4: Velocity and pressure at $t = 6.016$.

employed for the pressure the finite element $\mathbb{P}_1$. The linear fluid-structure system is solved using the LU decomposition.

After an initial transient period, the system settles into periodic oscilations, Figure 3. The average frequency in the time interval $[5, 8]$ is about 5.33 Hz. The results are similar to [5], where the reference amplitude of the periodic oscillations is 0.034, but the structure is a St. Venant-Kirchhoff material. The pressure in the structure domain has no physical signification and it is fixed to zero, Figure 4, at the right.

References

[1] Ciarlet PG. *Élasticité tridimensionnelle*, Masson, 1986.

[2] Hecht F. New development in FreeFem++. *J. Numer. Math.* 2012; **20**(3-4), 251–265, `http://www.freefem.org`.

[3] C.M. Murea, S. Sy, Updated Lagrangian/Arbitrary Lagrangian Eulerian framework for interaction between a compressible Neo-Hookean structure and an incompressible fluid, *Int. J. Numer. Meth. Engng.*, published online 17 June 2016, DOI: 10.1002/nme.5302

[4] S. Sy, C.M. Murea, Algorithm for solving fluid-structure interaction problem on a global moving mesh, *Coupled Systems Mechanics, An International Journal*, **1** (2012), No. 1, pp. 99–113

[5] Turek S, Hron J. Proposal for numerical benchmarking of fluid-structure interaction between an elastic object and laminar incompressible flow. In: *Fluid-Structure Interaction - Modelling, Simulation, Optimization*, Bungartz, H.-J., Schfer, M. (eds), Lect. Notes Comput. Sci. Eng., 53, Springer, Berlin, 2006; 371–385.