A monolithic semi-implicit algorithm for fluid-structure interaction problem at small structural displacements

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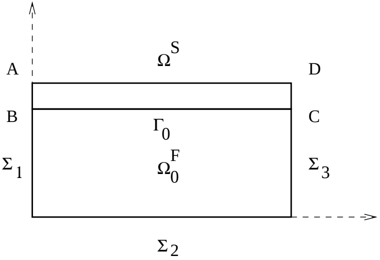
Initial (left) and intermediate (right) geometrical configuration



$$\partial\Omega_0^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_0, \qquad \partial\Omega_t^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_t.$$

Linear elasticity equations

We denote by $\mathbf{u}^S = \left(u_1^S, u_2^S\right)^T : \Omega_0^S \times [0, T] \to \mathbb{R}^2$ the structure displacement.

$$
\begin{aligned}
\rho^S \frac{\partial^2 \mathbf{u}^S}{\partial t^2} - \nabla \cdot \sigma^S &= \mathbf{f}^S, \quad \text{in } \Omega_0^S \times (0, T) \\
\sigma^S &= \lambda^S \left(\nabla \cdot \mathbf{u}^S\right) \mathbf{I} + 2\mu^S \epsilon \left(\mathbf{u}^S\right) \\
\epsilon\left(\mathbf{u}^S\right) &= \frac{1}{2}\left(\nabla \mathbf{u}^S + \left(\nabla \mathbf{u}^S\right)^T\right) \\
\mathbf{u}^S &= 0, \quad \text{on } \Gamma_D \times (0, T) \\
\sigma^S \mathbf{n}^S &= 0, \quad \text{on } \Gamma_N \times (0, T)
\end{aligned}
$$

$$\Gamma_D = [AB] \cup [CD], \quad \Gamma_N = [DA].$$

Navier-Stokes equations

We denote by $\mathbf{v}^F$ the fluid velocity and by p^F the fluid pressure.

$$
\begin{aligned}
\rho^F \left(\frac{\partial \mathbf{v}^F}{\partial t} + (\mathbf{v}^F \cdot \nabla)\mathbf{v}^F \right) - \nabla \cdot \sigma^F &= \mathbf{f}^F, \quad \forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F \\
\nabla \cdot \mathbf{v}^F &= 0, \quad \forall t \in (0, T), \forall \mathbf{x} \in \Omega_t^F \\
\sigma^F &= -p^F \mathbf{I} + 2\mu^F \epsilon \left(\mathbf{v}^F \right) \\
\sigma^F \mathbf{n}^F &= \mathbf{h}_{in}, \quad \text{on } \Sigma_1 \times (0, T) \\
\sigma^F \mathbf{n}^F &= \mathbf{h}_{out}, \quad \text{on } \Sigma_3 \times (0, T) \\
\mathbf{v}^F &= 0, \quad \text{on } \Sigma_2 \times (0, T)
\end{aligned}
$$

Interface and initial conditions

The interface Γ_t is the image of Γ_0 via the map

$$\mathbf{X} \rightarrow \mathbf{X} + \mathbf{u}^S(\mathbf{X}, t).$$

Interface conditions

$$\begin{aligned}
\mathbf{v}^F\left(\mathbf{X} + \mathbf{u}^S(\mathbf{X}, t), t\right) &= \frac{\partial \mathbf{u}^S}{\partial t}(\mathbf{X}, t), \ \forall (\mathbf{X}, t) \in \Gamma_0 \times (0, T) \\
\left(\sigma^F \mathbf{n}^F\right)_{\left(\mathbf{X}+\mathbf{u}^S(\mathbf{X},t),t\right)} &= -\left(\sigma^S \mathbf{n}^S\right)_{(\mathbf{X},t)}, \ \forall (\mathbf{X}, t) \in \Gamma_0 \times (0, T)
\end{aligned}$$

Initial conditions

$$\begin{aligned}
\mathbf{u}^S(\mathbf{X}, t = 0) &= \mathbf{u}^0(\mathbf{X}), \ \text{in } \Omega_0^S \\
\frac{\partial \mathbf{u}^S}{\partial t}(\mathbf{X}, t = 0) &= \dot{\mathbf{u}}^0(\mathbf{X}), \ \text{in } \Omega_0^S \\
\mathbf{v}^F(\mathbf{x}, t = 0) &= \mathbf{v}^0(\mathbf{x}), \ \text{in } \Omega_0^F
\end{aligned}$$

Arbitrary Lagrangian Eulerian (ALE) framework. Notations

The reference fluid domain $\widehat{\Omega}^F = \Omega_n^F$, the interface Γ_n
The velocity of the fluid mesh $\boldsymbol{\vartheta}^n = (\vartheta_1^n, \vartheta_2^n)^T$ is the solution of

$$\Delta \boldsymbol{\vartheta}^n = 0 \text{ in } \Omega_n^F, \quad \boldsymbol{\vartheta}^n = 0 \text{ on } \partial\Omega_n^F \setminus \Gamma_n, \quad \boldsymbol{\vartheta}^n = \mathbf{v}^{F,n} \text{ on } \Gamma_n$$

The ALE map $\mathcal{A}_{t_{n+1}} : \overline{\Omega}_n^F \to \mathbb{R}^2$

$$\mathcal{A}_{t_{n+1}}(\widehat{x}_1, \widehat{x}_2) = (\widehat{x}_1 + \Delta t \vartheta_1^n, \widehat{x}_2 + \Delta t \vartheta_2^n).$$

We define $\Omega_{n+1}^F = \mathcal{A}_{t_{n+1}}(\Omega_n^F)$ and $\Gamma_{n+1} = \mathcal{A}_{t_{n+1}}(\Gamma_n)$
We introduce $\widehat{\mathbf{v}}^{F,n+1} : \Omega_n^F \to \mathbb{R}^2$ and $\widehat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$ defined by

$$\widehat{\mathbf{v}}^{F,n+1}(\widehat{\mathbf{x}}) = \mathbf{v}^{F,n+1}(\mathbf{x}), \quad \widehat{p}^{F,n+1}(\widehat{\mathbf{x}}) = p^{F,n+1}(\mathbf{x}),$$

$\forall \widehat{\mathbf{x}} \in \Omega_n^F, \ \mathbf{x} = \mathcal{A}_{t_{n+1}}(\widehat{\mathbf{x}}) \in \Omega_{n+1}^F$

Time discretization of the fluid equations

Find $\widehat{\mathbf{v}}^{F,n+1} : \Omega_n^F \to \mathbb{R}^2$ and $\widehat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$ such that:

$$\rho^F \left(\frac{\widehat{\mathbf{v}}^{F,n+1} - \mathbf{v}^{F,n}}{\Delta t} + \left(\left(\mathbf{v}^{F,n} - \boldsymbol{\vartheta}^n \right) \cdot \nabla \right) \widehat{\mathbf{v}}^{F,n+1} \right)$$

$$-2\mu^F \nabla \cdot \epsilon \left(\widehat{\mathbf{v}}^{F,n+1} \right) + \nabla \widehat{p}^{F,n+1} = \widehat{\mathbf{f}}^{F,n+1}, \text{ in } \Omega_n^F$$

$$\nabla \cdot \widehat{\mathbf{v}}^{F,n+1} = 0, \text{ in } \Omega_n^F$$

$$\sigma^F(\widehat{\mathbf{v}}^{F,n+1}, \widehat{p}^{F,n+1}) \cdot \mathbf{n}^F = \mathbf{h}_{in}^{n+1}, \text{ on } \Sigma_1$$

$$\sigma^F(\widehat{\mathbf{v}}^{F,n+1}, \widehat{p}^{F,n+1}) \cdot \mathbf{n}^F = \mathbf{h}_{out}^{n+1}, \text{ on } \Sigma_3$$

$$\widehat{\mathbf{v}}^{F,n+1} = 0, \text{ on } \Sigma_2$$

$$\mathbf{v}^F(X,0) = \mathbf{v}^0(X), \text{ in } \Omega_0^F.$$

Time discretization of the structure equations

Find $\mathbf{u}^{S,n+1}$, $\dot{\mathbf{u}}^{S,n+1}$, $\ddot{\mathbf{u}}^{S,n+1}$: $\Omega_0^S \rightarrow \mathbb{R}^2$ such that:

$$\begin{aligned}
&\rho^S \ddot{\mathbf{u}}^{S,n+1} - \nabla \cdot \sigma^S(\mathbf{u}^{S,n+1}) = \mathbf{f}^{S,n+1}, \quad \text{in } \Omega_0^S \\
&\mathbf{u}^{S,n+1} = 0, \quad \text{on } \Gamma_D \\
&\sigma^S(\mathbf{u}^{S,n+1})\mathbf{n}^S = 0, \quad \text{on } \Gamma_N \\
&\mathbf{u}^S(X,0) = \mathbf{u}^0(X), \quad \text{in } \Omega_0^S
\end{aligned}$$

$$\begin{aligned}
\dot{\mathbf{u}}^{S,n+1} &= \dot{\mathbf{u}}^{S,n} + \Delta t\left[(1-\delta)\ddot{\mathbf{u}}^{S,n} + \delta\ddot{\mathbf{u}}^{S,n+1}\right] \\
\mathbf{u}^{S,n+1} &= \mathbf{u}^{S,n} + \Delta t\dot{\mathbf{u}}^{S,n} + (\Delta t)^2\left[\left(\frac{1}{2}-\theta\right)\ddot{\mathbf{u}}^{S,n} + \theta\ddot{\mathbf{u}}^{S,n+1}\right].
\end{aligned}$$

For $\delta = \frac{1}{2}$, the Newmark scheme is of second order in time.

Interface conditions

We define $\mathbb{T} = \mathcal{A}_{t_n} \circ \mathcal{A}_{t_{n-1}} \cdots \circ \mathcal{A}_{t_1}$.
We have $\mathbb{T}(\Gamma_0) = \Gamma_n$.

$$\begin{aligned} \widehat{\mathbf{v}}^{F,n+1} \circ \mathbb{T} &= \dot{\mathbf{u}}^{S,n+1}, \text{ on } \Gamma_0 \times (0,T] \\ (\sigma^F(\widehat{\mathbf{v}}^{F,n+1}, \widehat{p}^{F,n+1})\mathbf{n}^F) \circ \mathbb{T} &= -\sigma^S(\mathbf{u}^{S,n+1})\mathbf{n}^S, \text{ on } \Gamma_0 \times (0,T] \end{aligned}$$

Remark The global system of unknowns $\widehat{\mathbf{v}}^{F,n+1}$, $\widehat{p}^{F,n+1}$, $\mathbf{u}^{S,n+1}$, $\dot{\mathbf{u}}^{S,n+1}$, $\ddot{\mathbf{u}}^{S,n+1}$ is implicit, but the fluid domain is computed explicitly as the image of Ω_n^F via the map

$$\widehat{\mathbf{x}} \to \widehat{\mathbf{x}} + \Delta t\, \vartheta^n(\widehat{\mathbf{x}}).$$

This is the meaning of the term “semi-implicit” of the title.

Weak formulation of the fluid equations

$$\widehat{W}_n^F = \left\{ \widehat{\mathbf{w}}^F \in (H^1(\Omega_n^F))^2;\ \widehat{\mathbf{w}}^F = 0 \text{ on } \Sigma_2 \right\}, \quad \widehat{Q}_n^F = L^2(\Omega_n^F)$$

Find $\widehat{\mathbf{v}}^{F,n+1} \in \widehat{W}_n^F$ and $\widehat{p}^{F,n+1} \in \widehat{Q}_n^F$ such that:

$$\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}^F + \int_{\Omega_n^F} \rho^F \left(\left(\left(\mathbf{v}^{F,n} - \boldsymbol{\vartheta}^n \right) \cdot \nabla \right) \widehat{\mathbf{v}}^{F,n+1} \right) \cdot \widehat{\mathbf{w}}^F$$

$$- \int_{\Omega_n^F} \left(\nabla \cdot \widehat{\mathbf{w}}^F \right) \widehat{p}^{F,n+1} + \int_{\Omega_n^F} 2\mu^F \epsilon \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon \left(\widehat{\mathbf{w}}^F \right)$$

$$- \int_{\Gamma_n} \left(\sigma^F \mathbf{n}^F \right) \cdot \widehat{\mathbf{w}}^F = \mathcal{L}_F(\widehat{\mathbf{w}}^F), \quad \forall \widehat{\mathbf{w}}^F \in \widehat{W}_n^F$$

$$\int_{\Omega_n^F} \widehat{q}(\nabla \cdot \widehat{\mathbf{v}}^{F,n+1}) = 0, \quad \forall \widehat{q} \in \widehat{Q}_n^F$$

Weak formulation of the structure equations. Lagrangian coordinates

$$W^S = \left\{ \mathbf{w}^S \in (H^1(\Omega_0^S))^2; \;\; \mathbf{w}^S = 0 \;\; \text{on} \;\; \Gamma_D \right\}.$$

Find $\dot{\mathbf{u}}^{S,n+1} \in W^S$ such that:

$$\int_{\Omega_0^S} \frac{2\rho^S}{\Delta t} \dot{\mathbf{u}}^{S,n+1} \cdot \mathbf{w}^S + 2\theta \Delta t \, a_S(\dot{\mathbf{u}}^{S,n+1}, \mathbf{w}^S)$$
$$- \int_{\Gamma_0} (\sigma^S \mathbf{n}^S) \cdot \mathbf{w}^S = \mathcal{L}_S(\mathbf{w}^S), \quad \forall \mathbf{w}^S \in W^S,$$

where

$$a_S(\mathbf{u}, \mathbf{w}) = \int_{\Omega_0^S} \left[\lambda^S (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{w}) + 2\mu^S \epsilon(\mathbf{u}) : \epsilon(\mathbf{w}) \right]$$

Weak formulation of the structure equations. Eulerian coordinates

$$\widehat{W}_n^S = \left\{ \widehat{\mathbf{w}} \in (H^1(\Omega_n^S))^2; \quad \widehat{\mathbf{w}} = 0 \text{ on } \Gamma_D \right\}.$$

$$\int_{\Omega_n^S} \frac{2\rho^S}{\Delta t} \widehat{\mathbf{v}}^{S,n+1} \cdot \widehat{\mathbf{w}} + 2\theta\Delta t \; \tilde{a}_S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{\mathbf{w}})$$
$$- \int_{\Gamma_n} (\sigma^S \mathbf{n}^S) \cdot \widehat{\mathbf{w}} = \tilde{\mathcal{L}}_S(\widehat{\mathbf{w}}), \quad \forall \widehat{\mathbf{w}} \in \widehat{W}_n^S,$$

where

$$\tilde{a}_S(\mathbf{u}, \mathbf{w}) = \int_{\Omega_n^S} \left[\lambda^S (\nabla \cdot \mathbf{u})(\nabla \cdot \mathbf{w}) + 2\mu^S \epsilon(\mathbf{u}) : \epsilon(\mathbf{w}) \right]$$

Global moving domain

$$\Omega_n = \Omega_n^F \cup \Omega_n^S$$

Global velocity and pressure

$$\mathbf{v}^n : \Omega_n \to \mathbb{R}^2, \quad p^n : \Omega_n \to \mathbb{R}$$

$$\widehat{W}_n = \left\{ \hat{\mathbf{w}} \in (H^1(\Omega_n))^2;\ \hat{\mathbf{w}} = 0 \text{ on } \Gamma_D \cup \Sigma_2 \right\}, \quad \widehat{Q}_n = L^2(\Omega_n)$$

Characteristic functions related to the fluid domain $\chi_{\Omega_t^F} : \overline{\Omega}_t \to \mathbb{R}$ and structure domain $\chi_{\Omega_t^S} : \overline{\Omega}_t \to \mathbb{R}$:

$$\chi_{\Omega_t^S} = \begin{cases} 1, & \text{on } \overline{\Omega}_t^S \\ 0, & \text{otherwise} \end{cases} \qquad \chi_{\Omega_t^F} = 1 - \chi_{\Omega_t^S}.$$

Monolithic formulation for the fluid-structure equations

Find $(\widehat{\mathbf{v}}^{n+1}, \widehat{p}^{n+1}) \in \widehat{W}_n \times \widehat{Q}_n$ such that:

$$
\begin{aligned}
&\int_{\Omega_n} \chi_{\Omega_n^F}\, \rho^F \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}} + \int_{\Omega_n} \chi_{\Omega_n^F}\, \rho^F \left(\left(\left(\mathbf{v}^n - \tilde{\vartheta}^n \right) \cdot \nabla \right) \widehat{\mathbf{v}}^{n+1} \right) \cdot \widehat{\mathbf{w}} \\
&- \int_{\Omega_n} \chi_{\Omega_n^F} \, (\nabla \cdot \widehat{\mathbf{w}}) \, \widehat{p}^{n+1} + \int_{\Omega_n} \chi_{\Omega_n^F}\, 2\mu^F \epsilon \left(\widehat{\mathbf{v}}^{n+1} \right) : \epsilon \left(\widehat{\mathbf{w}} \right) \\
&+ \int_{\Omega_n} \chi_{\Omega_n^S} \frac{2\rho^S}{\Delta t} \widehat{\mathbf{v}}^{n+1} \cdot \widehat{\mathbf{w}} + 2\theta \Delta t \, \tilde{a}_S(\widehat{\mathbf{v}}^{n+1}, \widehat{\mathbf{w}}) \\
&= \tilde{\mathcal{L}}_F(\widehat{\mathbf{w}}) + \tilde{\mathcal{L}}_S(\widehat{\mathbf{w}}), \quad \forall \widehat{\mathbf{w}} \in \widehat{W}_n \\
&\int_{\Omega_n} \chi_{\Omega_n^F} \, \widehat{q} (\nabla \cdot \widehat{\mathbf{v}}^{n+1}) = 0, \quad \forall \widehat{q} \in \widehat{Q}_n .
\end{aligned}
$$

Finite element discretization

Triangular $\mathbb{P}_1 + \textit{bubble}$ for the velocity, $\mathbb{P}_1$ for the pressure and $\mathbb{P}_0$ for the characteristic functions.

The velocity, the pressure as well as the test functions are continuous all over the global domain Ω_n. Consequently, the continuity of velocity at the interface is automatically satisfied.

The integrals over the interface do not appear explicitly in the global weak form due to the action and reaction principle.

If the solution of the monolithic is sufficiently smooth, the continuity of stress at the interface holds in a weak sense.

Linear systems

The linear system has not an unique solution, because the pressure p^S can take any value.

$$\begin{bmatrix} A & B^T & 0 \\ B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ p^F \\ p^S \end{bmatrix} = \begin{bmatrix} \mathcal{L} \\ 0 \\ 0 \end{bmatrix}$$

We have added the term $\epsilon \int_{\Omega_n} \widehat{p}^{n+1} \widehat{q}$, then the bellow system has an unique solution and $p^S = 0$ on Ω_n^S.

$$\begin{bmatrix} A & B^T & 0 \\ B & \epsilon M^F & 0 \\ 0 & 0 & \epsilon M^S \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ p^F \\ p^S \end{bmatrix} = \begin{bmatrix} \mathcal{L} \\ 0 \\ 0 \end{bmatrix}$$

The matrix A is not symetric due to the convection term. We have used the GMRES algorithm for solving the linear system.

Time advancing schema from n to $n+1$

We assume that we know Ω_n, $\mathbf{v}^n$, $\mathbf{u}^n$, $\ddot{\mathbf{u}}^n$.

Step 1: Compute $\tilde{\boldsymbol{\vartheta}}^n$

Step 2: Solve the linear system and get $\widehat{\mathbf{v}}^{n+1}$ and pressure $\widehat{p}^{n+1}$

Step 3: Compute the displacement and the acceleration

$$
\begin{aligned}
\widehat{\mathbf{u}}^{n+1} &= \mathbf{u}^n + (\Delta t)^2\left(\frac{1}{2} - 2\theta\right)\ddot{\mathbf{u}}^n + \Delta t(1-2\theta)\mathbf{v}^n + 2\theta\Delta t\,\widehat{\mathbf{v}}^{n+1} \\
\widehat{\ddot{\mathbf{u}}}^{n+1} &= \frac{2}{\Delta t}\left(\widehat{\mathbf{v}}^{n+1} - \mathbf{v}^n\right) - \ddot{\mathbf{u}}^n
\end{aligned}
$$

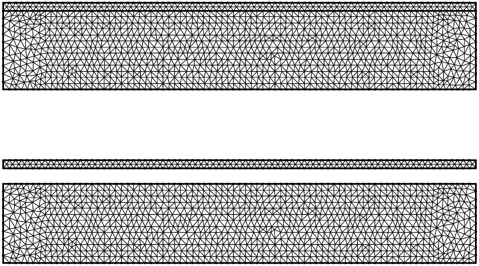
Step 4: We define the map $\mathbb{T}_n : \overline{\Omega}_n \to \mathbb{R}^2$ by:

$$
\mathbb{T}_n(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + \Delta t \tilde{\boldsymbol{\vartheta}}^n(\widehat{\mathbf{x}})\chi_{\Omega_n^F}(\widehat{\mathbf{x}}) + (\widehat{\mathbf{u}}^{n+1}(\widehat{\mathbf{x}}) - \mathbf{u}^n(\widehat{\mathbf{x}}))\chi_{\Omega_n^S}(\widehat{\mathbf{x}})
$$

Step 5: We set $\Omega_{n+1} = \mathbb{T}_n(\Omega_n)$. We define $\mathbf{v}^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ and $p^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ by:

$$
\mathbf{v}^{n+1}(\mathbf{x}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}),\;\; p^{n+1}(\mathbf{x}) = \widehat{p}^{n+1}(\widehat{\mathbf{x}}),\;\; \forall\widehat{\mathbf{x}} \in \Omega_n \text{ and } \mathbf{x} = \mathbb{T}_n(\widehat{\mathbf{x}}).
$$

Global fluid-structure mesh (top), the structure and fluid meshes (bottom)



The global moving mesh is compatible with the interface: a triangle of the global mesh belongs either to the fluid region or to the structure region.

CPU time: monolithic versus partionned procedure algorithm

$nsFS$	nt	nv	global Dof	CPU_{mono}
80	2426	1305	8767	5m25s
100	3916	2070	14042	7m18s
120	5039	2649	18016	11m34s

$nsSF$	ntF	nvF	ntS	nvS	DofF	DofS	CPU_{pp}	$\frac{CPU_{pp}}{CPU_{mono}}$
80	2106	1144	320	242	7644	484	10m49s	1.99
100	3538	1880	378	291	12716	582	15m57s	2.18
120	4582	2422	452	348	16430	696	21m06s	1.82

m	CPU_{pp}	$\frac{CPU_{pp}}{CPU_{mono}}$
3	10m49s	1.99
7	27m47s	5.12
10	41m07s	7.59

Conclusions

- Semi-implicit algorithm: the global system of unknowns $\widehat{\mathbf{v}}^{n+1}$, $\widehat{p}^{n+1}$ is implicit, but the fluid domain is computed explicitly.
- The continuity of velocity at the interface is automatically satisfied and the continuity of stress holds in a weak sense.
- The global system is solved monolithically.
- The characteristic functions permit us to choose independently the time discretization schemes of the fluid and structure.
- The global moving mesh is obtained by gluing the fluid and structure meshes which are matching at the interface. The interface does not pass through the triangles.
- The CPU time is reduced compared to a particular partition procedures strategy.