

# A Penalization Method for the Elliptic Bilateral Obstacle Problem

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**Abstract.** In this paper we propose a new algorithm for the wellknown elliptic bilateral obstacle problem. Our approach enters the category of fixed domain methods and solves just linear elliptic equations at each iteration. The approximating coincidence set is explicitly computed. In the numerical examples, the algorithm has a fast convergence.

**Keywords:** obstacle problem, free boundary problems, penalization

## 1 Introduction

The obstacle problem may be formulated as an elliptic variational inequality. Detailed theoretical discussions of various variational inequalities may be found in [4], [15], [23]. Applications, including optimal control problems are investigated in the books [1], [5], [6], [24]. From the point of view of the numerical approximation, we quote just the monographs [6], [7], [21].

In this paper we propose an algorithm for the elliptic bilateral obstacle problem which is of fixed domain type in the sense that the finite element discretization is given in the whole domain, independently of the position of the unknown free boundary. In each iteration a linear elliptic equation has to be solved in the whole domain and the corresponding stiffness matrix is common for all iterations. This is a clear advantage from the point of view of the implementation and the approximating coincidence set is explicitly computed in each iteration and it converges in the Hausdorff-Pompeiu sense [20] to the searched geometry. Moreover, we need just a scalar penalization parameter in our method. A similar strategy was employed in [18] for the elliptic unilateral obstacle problem and for parabolic variational inequalities.

Our approach is inspired from shape optimization techniques, but no shape optimization problem is used here although this is a known method in free boundary problems, [2]. One may compare the present approach to the recent works

[8], [19], [22]. An efficient Lagrangian method together with a primal-dual active set strategy with regularization is studied in [13]. But our approach and arguments are certainly different. We also quote the multi grid method employed in [12], the path-following method for semi-smooth Newton schemes [10] and a duality-type method [16], [17].

## 2 Formulation of the Problem and the Algorithm

Let  $D$  be a smooth domain in  $\mathbb{R}^d$ ,  $d \in \mathbb{N}^*$  and  $f \in L^2(D)$  be given. We denote the obstacles by  $\psi_1, \psi_2 : D \rightarrow \mathbb{R}$ , such that  $\psi_1, \psi_2 \in H^2(D)$ ,  $\psi_1 \leq \psi_2$  in  $D$ ,  $\psi_1|_{\partial D} \leq 0$ ,  $\psi_2|_{\partial D} \geq 0$ . The admissible set

$$K = \{v \in H_0^1(D); \psi_1(x) \leq v(x) \leq \psi_2(x) \text{ a.e. in } D\}$$

is a nonvoid closed convex subset of  $H_0^1(D)$ .

To  $K$ , the following variational inequality, may be associated:

$$\int_D \nabla y \cdot (\nabla y - \nabla v) dx \leq \int_D f(y - v) dx, \quad \forall v \in K. \quad (1)$$

The existence of a unique solution  $y \in K$  is wellknown.

We introduce  $\beta \subset \mathbb{R} \times \mathbb{R}$  the maximal monotone graph given by

$$\beta(r) = \begin{cases} \emptyset, & r < 0, \\ ] -\infty, 0], & r = 0, \\ 0, & r > 0, \end{cases} \quad (2)$$

the maximal monotone graph  $\gamma \subset \mathbb{R} \times \mathbb{R}$  given by

$$\gamma(r) = \begin{cases} 0, & r < 0, \\ [0, +\infty[, & r = 0, \\ \emptyset, & r > 0 \end{cases} \quad (3)$$

and denote by  $\beta_\epsilon, \gamma_\epsilon$ ,  $\epsilon > 0$ , their Yosida approximations. We have

$$\beta_\epsilon(r) = \begin{cases} \frac{1}{\epsilon}r, & r \leq 0, \\ 0, & r > 0, \end{cases} \quad \gamma_\epsilon(r) = \begin{cases} 0, & r < 0, \\ \frac{1}{\epsilon}r, & r \geq 0. \end{cases}$$

Notice that  $\beta_\epsilon$  is a concave and  $\gamma_\epsilon$  is a convex function in  $\mathbb{R}$ .

In the case when  $f \in L^2(D)$ ,  $\psi_1, \psi_2 \in H^2(D)$  with the compatibility condition  $\psi_1 \leq \psi_2$  in  $D$ ,  $\psi_1|_{\partial D} \leq 0$ ,  $\psi_2|_{\partial D} \geq 0$ , it is known that the solution of (1) satisfies the regularity property  $y \in H^2(D)$ . Moreover, in this case, the obstacle problem may be written as a multivalued equation

$$-\Delta y + \beta(y - \psi_1) + \gamma(y - \psi_2) \ni f \text{ in } D. \quad (4)$$

One can define two coincidence sets, corresponding to the two obstacles:

$$\begin{aligned} D_1 &= \{x \in D; y(x) = \psi_1(x)\} \\ D_2 &= \{x \in D; y(x) = \psi_2(x)\} \end{aligned}$$

and associated to (1).

We state now our algorithm.

**Algorithm**

1) Choose  $n = 0$ ,  $\epsilon_0 > 0$ ,  $\Omega_1^0 \subset D$ ,  $\Omega_2^0 \subset D$  open subsets such that  $(D \setminus \Omega_1^0) \cap (D \setminus \Omega_2^0) = \emptyset$ ,  $\tilde{y}_{-1} = 0$ ;

2) Compute  $y_n \in H_0^1(D)$  as solution of the linear elliptic equation

$$-\Delta y_n + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_1^n} (y_n - \psi_1) + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_2^n} (y_n - \psi_2) = f \text{ in } D \quad (5)$$

3) Compute  $\mathbf{y}_n = \min \{\psi_2, \max\{y_n, \psi_1\}\}$ ,  $\Omega_1^{n+1} = \{x \in D; \mathbf{y}_n(x) > \psi_1(x)\}$ ,  $\Omega_2^{n+1} = \{x \in D; \mathbf{y}_n(x) < \psi_2(x)\}$   $\epsilon_{n+1} = \frac{\epsilon_n}{2}$ ;

4) If  $\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_{H^1(D)} < tol$  then STOP else  $n=n+1$  GO TO step 2.

*Remark 1.* By the classical result of [3], the elastic-plastic torsion problem is equivalent with a variational inequality of obstacle type and our algorithm may be applied as well.

We convene to extend the value  $\frac{1}{\epsilon}$  for  $\beta'_\epsilon$  and  $\gamma'_\epsilon$  in the origin and we can rewrite the step 2 of the Algorithm as

$$-\Delta y_n + (\beta'_{\epsilon_n} (y_{n-1} - \psi_1)) (y_n - \psi_1) + (\gamma'_{\epsilon_n} (y_{n-1} - \psi_2)) (y_n - \psi_2) = f. \quad (6)$$

Recall that the usual approximation by regularization of the variational inequality (1) is

$$-\Delta \tilde{y}_n + \beta_{\epsilon_n} (\tilde{y}_n - \psi_1) + \gamma_{\epsilon_n} (\tilde{y}_n - \psi_2) = f \text{ in } D, \quad (7)$$

plus homogeneous boundary conditions on  $\partial D$ .

Notice that  $\beta_\epsilon(r) = \beta'_\epsilon(r)r$  and  $\gamma_\epsilon(r) = \gamma'_\epsilon(r)r$ , under the above convention, which shows that (6) and (7) have very similar structure. Clearly, (7) is a non-linear elliptic equation, while the decoupling operated in (6) allows to use linear elliptic equations.

### 3 Stability

We present in this section a stability result in  $L^2(D)$  for the algorithm introduced above applied to the bilateral obstacle problem.

**Theorem 1.** *i) The sequence  $\{y_n\}$  is bounded in  $L^2(D)$ .*

*ii) There is  $C > 0$ , independent of  $n$ , such that:*

$$\int_{D \setminus \Omega_2^n} (y_n - \psi_2)_+^2 dx + \int_{D \setminus \Omega_1^n} (y_n - \psi_1)_-^2 dx \leq C \epsilon_n. \quad (8)$$

*Proof.* Using  $\beta_\epsilon(r) = \beta'_\epsilon(r)r$ , the concavity of  $\beta_\epsilon(\cdot)$  and the definition of the subdifferential of concave mapping, we obtain

$$\begin{aligned} & (\beta'_{\epsilon_n}(y_{n-1} - \psi))(y_n - \psi) = (\beta'_{\epsilon_n}(y_{n-1} - \psi))(y_{n-1} - \psi) \\ & + (\beta'_{\epsilon_n}(y_{n-1} - \psi))(y_n - \psi - y_{n-1} + \psi) \geq \beta_{\epsilon_n}(y_{n-1} - \psi) \\ & + \beta_{\epsilon_n}(y_n - \psi) - \beta_{\epsilon_n}(y_{n-1} - \psi) = \beta_{\epsilon_n}(y_n - \psi). \end{aligned}$$

We use the above inequality in the equation (5). We get

$$-\Delta y_n + \beta_{\epsilon_n}(y_n - \psi_1) + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_2^n}(y_n - \psi_2) \leq f, \quad (9)$$

where  $\beta$  is given by (2) and  $\beta_{\epsilon_n}$  is its regularization.

We multiply (9) by  $(y_n - \psi_2)_+$  and we use that

$$\beta_{\epsilon_n}(y_n - \psi_1)(y_n - \psi_2)_+ = 0. \quad (10)$$

While  $\beta_{\epsilon_n}(y_n - \psi_1)$  may take negative values, this happens for  $y_n \leq \psi_1$ , that is  $y_n - \psi_2 \leq 0$  (since  $\psi_1 \leq \psi_2$ ). Then  $(y_n - \psi_2)_+ = 0$  and (10) follows. We infer

$$\begin{aligned} & \int_D |\nabla(y_n - \psi_2)_+|^2 + \frac{1}{\epsilon_n} \int_{D \setminus \Omega_2^n} (y_n - \psi_2)_+^2 \\ & \leq \int_D f(y_n - \psi_2)_+ + \int_D \nabla \psi_2 \cdot \nabla(y_n - \psi_2)_+. \end{aligned} \quad (11)$$

By the conditions  $\psi_2|_{\partial D} \geq 0$  and  $y_n|_{\partial D} = 0$ , we have  $(y_n - \psi_2)_+ = 0$  on  $\partial D$  and the Poincaré inequality shows that  $\{(y_n - \psi_2)_+\}$  is bounded in  $H_0^1(D)$ , by (11).

Equation (5) may be rewritten in the form

$$-\Delta y_n + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_1^n}(y_n - \psi_1) + \gamma'_{\epsilon_n}(y_{n-1} - \psi_2)(y_n - \psi_2) = f. \quad (12)$$

We compute

$$\begin{aligned} & \gamma'_{\epsilon_n}(y_{n-1} - \psi_2)(y_n - \psi_2) = \gamma'_{\epsilon_n}(y_{n-1} - \psi_2)(y_{n-1} - \psi_2) \\ & + \gamma'_{\epsilon_n}(y_{n-1} - \psi_2)(y_n - \psi_2 - y_{n-1} + \psi_2) = \gamma_{\epsilon_n}(y_{n-1} - \psi_2) \\ & + \gamma'_{\epsilon_n}(y_{n-1} - \psi_2)(y_n - \psi_2 - y_{n-1} + \psi_2) \leq \gamma_{\epsilon_n}(y_{n-1} - \psi_2) \\ & + \gamma_{\epsilon_n}(y_n - \psi_2) - \gamma_{\epsilon_n}(y_{n-1} - \psi_2) = \gamma_{\epsilon_n}(y_n - \psi_2) \end{aligned} \quad (13)$$

using the subdifferential of convex mappings.

By (12), (13), we obtain

$$-\Delta y_n + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_1^n}(y_n - \psi_1) + \gamma_{\epsilon_n}(y_n - \psi_2) \geq f. \quad (14)$$

Multiply (14) by  $-(y_n - \psi_1)_- \in H_0^1(D)$ , due to  $y_n = 0$  on  $\partial D$ ,  $\psi_1 \leq 0$  on  $\partial D$ :

$$\begin{aligned} & \int_D |\nabla(y_n - \psi_1)_-|^2 + \frac{1}{\epsilon_n} \int_{D \setminus \Omega_1^n} (y_n - \psi_1)_-^2 - \gamma_{\epsilon_n}(y_n - \psi_2)(y_n - \psi_1)_- \\ & \leq - \int_D f(y_n - \psi_1)_- + \int_D \nabla \psi_1 \cdot \nabla(y_n - \psi_1)_-. \end{aligned} \quad (15)$$

Notice that

$$-\gamma_{\epsilon_n}(y_n - \psi_2)(y_n - \psi_1)_- = 0 \quad (16)$$

since  $-\gamma_{\epsilon_n}(y_n - \psi_2)$  may take negative values just for  $y_n \geq \psi_2 \geq \psi_1$  and in this case  $(y_n - \psi_1)_- = 0$ . By (15), (16) we obtain:

$$\begin{aligned} & \int_D |\nabla(y_n - \psi_1)_-|^2 + \frac{1}{\epsilon_n} \int_{D \setminus \Omega_1^+} (y_n - \psi_1)_-^2 \\ & \leq - \int_D f(y_n - \psi_1)_- + \int_D \nabla \psi_1 \cdot \nabla(y_n - \psi_1)_-. \end{aligned} \quad (17)$$

Relation (17) shows that  $\{(y_n - \psi_1)_-\}$  is bounded in  $H_0^1(D)$ , by the Poincaré inequality.

We use the inequality

$$(x - b)_+ \leq (x - a)_+ + (a - b)_+$$

and we have

$$(y_n - \psi_1)_+ \leq (y_n - \psi_2)_+ + (\psi_2 - \psi_1)_+ = (y_n - \psi_2)_+ + \psi_2 - \psi_1.$$

Relation (11) shows that  $\{(y_n - \psi_1)_+\}$  is bounded in  $L^2(D)$ . In combination with (17), it yields  $\{y_n\}$  bounded in  $L^2(D)$ .

Relation (8) follows by adding (11) and (17) and using the already established boundedness of all the terms except the penalization term. This ends the proof.

*Remark 2.* In fact, the above proof shows that  $\{y_n\}$  bounded in  $L^p(D)$ ,  $p > 2$  depending on the dimension of  $D$ . Relation (8) says that the sequence  $\{y_n\}$  does not overpass the obstacles  $\psi_1, \psi_2$ , in the limit. The proof also provides partial information on  $\{\nabla y_n\}$ , but it is unclear whether  $\{y_n\}$  is bounded in  $H_0^1(D)$ .

## 4 Numerical Tests

We have used the software FreeFem++ v 3.19, [9]. For all tests, we use the same initial guess for the coincidence set  $D \setminus \Omega_1^0 = \emptyset$  and  $D \setminus \Omega_2^0 = \emptyset$ .

### Test 1.

We consider the torsion of an elastic-plastic prism studied in [7] p. 133 and [25]. The cross-section of the prism is  $D = [0, 1] \times [0, 1]$ . We solve the problem (1) with  $f(x) = -8$  where  $K = \{v \in H_0^1(D); -1 \leq \nabla v(x) \leq 1 \text{ a.e. in } D\}$ . For  $v \in K$ , the set  $\{x \in D; |\nabla v(x)| < 1\}$  is the elastic zone and  $\{x \in D; |\nabla v(x)| = 1\}$  is the plastic zone, [5], p. 264. By the result of [3], the elastic-plastic torsion problem is equivalent with a variational inequality of obstacle type  $K = \{v \in H_0^1(D); \psi_1(x) \leq v(x) \leq \psi_2(x) \text{ a.e. in } D\}$  where  $\psi_1(x) = -\text{dist}(x, \partial D)$  and  $\psi_2(x) = \text{dist}(x, \partial D)$ . If  $f < 0$ , then  $y \leq 0$ , consequently the top obstacle will be inactive.

We use a mesh of 39158 triangles, 19836 vertices and size  $h = \frac{1}{128}$ . The tolerance for the stopping test is  $tol = 10^{-3}$  and the penalization parameter is



**Fig. 1.** Test 1. The computed coincidence set of the plastic zone for the bottom obstacle at the first (left), second (middle) and last (right) iteration.

$\epsilon_n = 0.003$ . The coincidence set of the solution presented in Figure 1 at the right is similar to the above references.

Our algorithm stops after 6 iterations and the relative error in the  $H^1$  norm at the last iteration is  $\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_{H^1(D)} = 1.5 \times 10^{-5}$ .

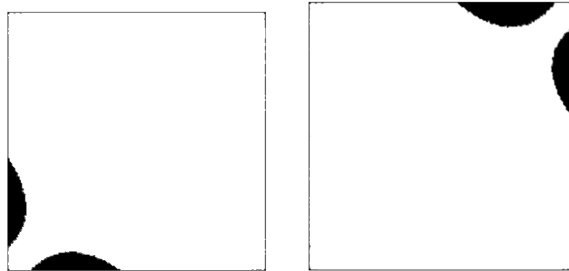
In [18], we have tested numerically with positive results the stability of a similar algorithm when  $f$  the right-hand side in (1) is perturbed.

**Test 2.**

We solve the problem (1) where

$$K = \{v \in H_0^1(D); \psi_1(x) \leq v(x) \leq \psi_2(x) \text{ a.e. in } D\},$$

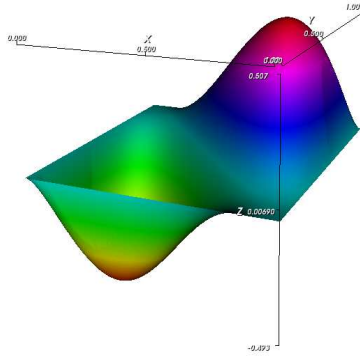
$D = [0, 1] \times [0, 1]$ ,  $\psi_1(x) = -\text{dist}(x, \partial D)$ ,  $\psi_2(x) = \text{dist}(x, \partial D)$  and  $f(x) = 11(x + y - 1)$ . Now both obstacles are active.



**Fig. 2.** Test 2. Coincidence sets of the plastic zone for the bottom obstacle (left) and for the top obstacle (right).

We use a mesh of 39158 triangles, 19836 vertices, the size  $h = \frac{1}{128}$ , the tolerance for the stopping test  $tol = 10^{-3}$  and the penalization parameter is  $\epsilon_n = 0.003$ . The algorithm stops in 4 iterations and the relative error in the  $H^1$  norm at the last iteration is  $\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_{H^1(D)} = 0.000209$ .

The coincidence sets are presented in Figure 2 and the computed solution in Figure 3.



**Fig. 3.** Test 2. Computed solution.

We solved the problem on different meshes, see Table 1. We denote by  $\mathbf{u}_i$ , the solution obtained using the mesh no.  $i$ .

mesh no.	mesh size $h$	triangles	vertices	$\ \mathbf{u}_i - \mathbf{u}_{i-1}\ _{H^1(D)}$
1	1/64	9720	4989	-
2	1/128	39158	19836	0.019005
3	1/256	154050	77538	0.008826
4	1/512	630326	316188	0.005768

**Table 1.** Test 2. Mesh parameters.

In [11], for the semi-smooth Newton method, it is proved a mesh-independence result: the continuous and the discrete process, converge q-linearly with the same rate.

**Test 3.**

Now we test the algorithm for the torsion of the elastic-plastic prism discussed in [14] and [26]. We can put this problem in the form (1). Let  $D = [0, 1] \times [0, 1]$ ,

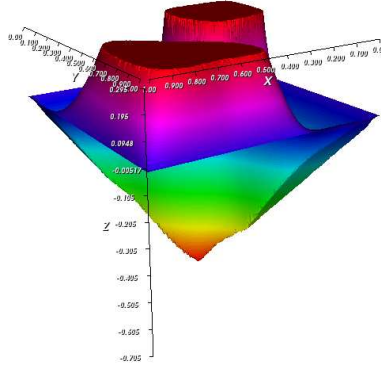
$\psi_1(x, y) = -\text{dist}((x, y), \partial D)$ ,  $\psi_2(x, y) = 0.2$  for all  $(x, y) \in D$  and set

$$g(x) = \begin{cases} 6x, & 0 < x \leq 1/6, \\ 2(1 - 3x), & 1/6 < x \leq 1/3, \\ 6(x - 1/3), & 1/3 < x \leq 1/2, \\ 2(1 - 3(x - 1/3)), & 1/2 < x \leq 2/3, \\ 6(x - 2/3), & 2/3 < x \leq 5/6, \\ 2(1 - 3(x - 2/3)), & 5/6 < x \leq 1 \end{cases}$$

and

$$f(x, y) = \begin{cases} 300, & (x, y) \in S = \{(x, y) \in D; |x - y| \leq 0.1 \text{ \& } x \leq 0.3\}, \\ -70 \exp(y)g(x), & x \leq 1 - y \text{ and } (x, y) \notin S, \\ 15 \exp(y)g(x), & x > 1 - y \text{ and } (x, y) \notin S. \end{cases}$$

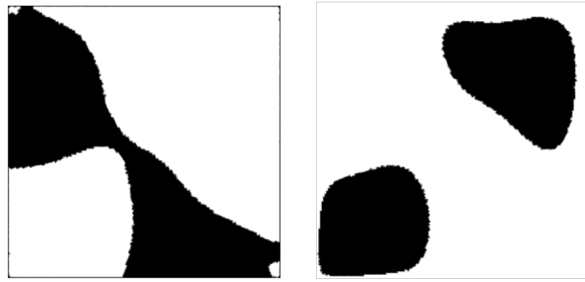
We use a mesh of 39158 triangles, 19836 vertices, the size  $h = \frac{1}{128}$ , the tolerance for the stopping test  $\text{tol} = 10^{-3}$  and the penalization parameter is  $\epsilon_n = 0.03$ . The computed solution after 6 iterations is presented in Figure 4 and the corresponding coincidence sets in Figure 5. The relative error in the  $H^1$  norm at the last iteration is  $\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_{H^1(D)} = 2.7 \times 10^{-5}$ .



**Fig. 4.** Test 3. Computed solution.

In [14], an augmented lagrangian active set strategy is employed. At each iteration, a reduced linear system associated with the inactive set is solved. In [26], at each iteration, linear systems associated to the complementary of the coincidence sets are solved.





**Fig. 5.** Test 3. Coincidence sets for the bottom obstacle (left) and for the top obstacle (right).

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