Fluid structure interaction with a reduced number of parameters

C. M. Murea
Laboratoire de Mathématiques et Applications,
Université de Haute-Alsace,
4, rue des Frères Lumière,
68093 MULHOUSE Cedex, France,
e-mail: c.murea@uha.fr
http://www.math.uha.fr/~edp/murea/

1 Introduction

In this paper we consider a two dimensional fluid structure interaction. The mathematical model which governs the fluid is the steady Stokes equations, while the structure verifies the beam equation without shearing stress. The solution of the model is given by the displacement of the structure, the velocity and the pressure of the fluid. The boundary of the fluid can be decomposed into a moving part, which is the interface between the fluid and the structure, and a rigid part.

The existence results for the fluid structure interaction can be funded in [6] for the steady case and in [5], [2] for the unsteady case.

The asymptotic limit when the fluid domain width tends to zero can be modelled by a one dimensional approach of Stokes equation, widely used in lubrication theory (see [1]).

In a previous work [9], a three dimensional fluid structure interaction was formulated as an optimal control system, where the control is the force acting on the interface and the observation is the velocity of the fluid on the interface. The fluid equations were solved with given surface force on the interface.

A similar approach was used in [10], where it was proved that the cost function is differentiable. The analytic computation of the gradient for the cost function is important in order to apply accurate numerical methods (see [8]). The exact gradient of the cost function is computed in [10].

In [11] are presented numerical results for a two dimensional fluid structure interaction using the optimal control method. The fluid equations are to be solved subject to the conditions of zero normal velocity and a given value of pressure on the interface. The control is the value of the pressure at the interface and the observation is the tangential velocity on the interface.

In the present work, a fluid structure interaction problem was formulated as an optimal control system, where the control is the force acting on the interface and the observation is the pressure on the interface. The boundary condition to be imposed on the fluid is that all components of the velocity are zero at the interface.

First, we prove that the normal force acting on the structure depends only on the pressure. Then, an explicit solution for a particular fluid structure interaction is given. Finally, numerical results are presented, where the control is an affine function.

2 Notations

Let L and H be two positive constants. For a given $e \in (0, H)$ we define the set

$$\begin{aligned} \mathcal{U}_{ad} &= \left\{ u \in \mathcal{C}^1 ([0, L]) ; \, u(0) = u(L) = u'(0) = u'(L) = 0, \right. \\ &\quad \left. H + u(x_1) \geq e, \, \forall x_1 \in [0, L] \right\} \end{aligned}$$

where u' is the first derivative of u.

For each $u \in \mathcal{U}_{ad}$, we introduce the notations (see the Figure 1)

$$\begin{aligned} \Omega_u^F &= \left\{ (x_1, x_2) \in \mathbb{R}^2 ; \, x_1 \in (0, L), \, 0 < x_2 < H + u(x_1) \right\}, \\ \Gamma_u &= \left\{ (x_1, x_2) \in \mathbb{R}^2 ; \, x_1 \in (0, L), \, x_2 = H + u(x_1) \right\}. \end{aligned}$$

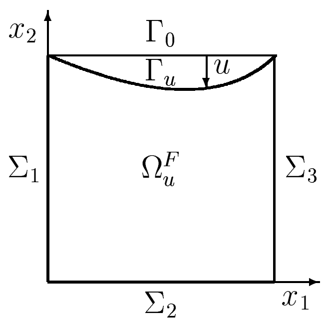

Figure 1: Sets appearing in the fluid-structure problem

Also, we denote

$$\begin{aligned}
\Sigma_1 &= \left\{(0,x_2)\in\mathbb{R}^2;\, x_2\in(0,H)\right\}\\
\Sigma_2 &= \left\{(x_1,0)\in\mathbb{R}^2;\, x_1\in(0,L)\right\}\\
\Sigma_3 &= \left\{(L,x_2)\in\mathbb{R}^2;\, x_2\in(0,H)\right\}.
\end{aligned}$$

The two-dimensional domain occupied by the fluid is Ω_u^F, the interface between the fluid and the structure is Γ_u, while $\Sigma = \Sigma_1\cup\overline{\Sigma_2}\cup\Sigma_3$ represents the rigid boundary of the fluid.

In the following, we denote by $n=(n_1,n_2)^T$ the unit outward normal vector and by $\tau=(\tau_1,\tau_2)^T=(-n_2,n_1)^T$ the unit tangential vector to $\partial\Omega_u^F$.

3 Position of the problem

We suppose that the fluid is governed by the steady Stokes equations, while the deformation of the elastic part of the boundary verifies a particular beam equation without shearing stress (see [3]). We consider that the structure is a beam of axis parallel to Ox_1 and with a constant cross section S. We assume that the displacement of the beam is normal to its axis.

The problem is to find:

- $u:[0,L]\to\mathbb{R}$ the displacement of the structure,
- $v=(v_1,v_2)^T:\Omega_u^F\to\mathbb{R}^2$ the velocity of the fluid and
- $p:\Omega_u^F\to\mathbb{R}$ the pressure of the fluid,

such that

$$\begin{aligned}
u''''(x_1) &= \frac{1}{D}\left(f^S(x_1)+p\left(x_1,H+u(x_1)\right)\right), \quad \forall x_1\in(0,L) && (1)\\
u\left(0\right) &= u\left(L\right)=u'\left(0\right)=u'\left(L\right)=0 && (2)\\
\max_{x_1\in[0,L]}|u(x_1)| &\le H-e && (3)\\
-\mu\Delta v+\nabla p &= f^F, \quad in\ \Omega_u^F && (4)\\
\text{div } v &= 0, \quad in\ \Omega_u^F && (5)\\
v &= g, \quad on\ \Sigma && (6)\\
v &= 0, \quad on\ \Gamma_u && (7)
\end{aligned}$$

where

- $D = E \cdot I_3$ is the rigidity to bending modulus for the axis Ox_3, E is the Young modulus, I_3 is the inertia momentum of the section with respect to the axis Ox_3.

- $f^S : (0, L) \to \mathbb{R}$ are the averaged volume forces of the structure, in general the gravity forces,

- $\mu > 0$ is the viscosity of the fluid,

- $f^F = (f_1^F, f_2^F)^T : \Omega_u^F \to \mathbb{R}^2$ are the volume forces of the fluid, in general the gravity forces,

- $g = (g_1, g_2)^T : \Sigma \to \mathbb{R}^2$ is the imposed velocity profile of the fluid on the rigid boundary, such that $\int_\Sigma g \cdot n \, d\sigma = 0$.

For the Newtonian fluids, the stress tensor σ has the form

$$\sigma = -p\, I + \mu \left(\nabla v + \nabla v^T \right),$$

consequently, the fluid forces acting on the structure are $-\sigma n$.

Proposition 1 *If* $v \in \left(H^2 \left(\Omega_u^F \right) \right)^2$, $p \in H^1 \left(\Omega_u^F \right)$, v *is constant on* Γ_u, $\operatorname{div} v = 0$ *in* Ω_u^F, *then* $-(\sigma n) \cdot n = p$ *on* Γ_u.

Proof. This result is a corollary of the Proposition 3.1 from [12] and it is similar to the Proposition 4.5 from the same paper. $\Box$

Under the assumption of small displacement of the beam, it follows that $n \approx (0, 1)^T$. Then, it is reasonable to solve the beam equation (1) under the fluid forces given by $p\,(x_1, H + u(x_1))$, $x_1 \in (0, L)$.

4 Optimal control approach

For given α and β two real parameters, we can solve the beam equations

$$u''''(x_1) = \frac{1}{D} \left(f^S(x_1) - \alpha - \beta\, x_1 \right), \quad \forall x_1 \in (0, L) \tag{8}$$

$$u\,(0) = u\,(L) = u'\,(0) = u'\,(L) = 0 \tag{9}$$

If $u \in \mathcal{U}_{ad}$, we can solve the Stokes equations (4)–(7) and we obtain v and p. The pressure is determined up to an additive constant, i.e. it has the form $p = p_0 + C$, where p_0 is a particular solution and C is a real constant.

We will take p such that

$$\int_{\Gamma_u} (p+\alpha+\beta\, x_1)^2\, d\sigma = \inf_{C\in\mathbb{R}} \int_{\Gamma_u} (p_0+C+\alpha+\beta\, x_1)^2\, d\sigma \tag{10}$$

The optimal C is given by the formula

$$C = -\frac{\int_{\Gamma_u} (p_0+\alpha+\beta\, x_1)\, d\sigma}{\int_{\Gamma_u} d\sigma}$$

and we have

$$\int_{\Gamma_u} (p+\alpha+\beta\, x_1)^2\, d\sigma = \frac{\int_{\Gamma_u} d\sigma \int_{\Gamma_u} (p_0+\alpha+\beta\, x_1)^2\, d\sigma - \left(\int_{\Gamma_u} (p_0+\alpha+\beta\, x_1)\ d\sigma\right)^2}{\int_{\Gamma_u} d\sigma}$$

Let $J : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$J(\alpha,\beta) = \int_{\Gamma_u} (p-\alpha-\beta x_1)^2\, d\sigma$$

Now, the problem is to find $(\alpha,\beta)\in\mathbb{R}^2$ solution of

$$\left\{\begin{array}{l} \inf J(\alpha,\beta) \\ u \textit{ solution of } (8)-(9) \\ \max_{x_1\in[0,L]} |u(x_1)| \le H-e \\ v,\ p \textit{ solution of } (4)-(7) \\ p \textit{ verifies } (10) \end{array}\right. \tag{11}$$

In other words, we try to find a solution of the system (1)–(7) such that

$$p\,(x_1, H+u(x_1)) \approx -\alpha-\beta\, x_1, \quad \forall x_1\in(0,L)$$

where α and β are two real parameters.

5 Explicit solution for a particular case

We assume that the density of the fluid is constant ρ^F and the volume forces in the fluid have the form $f^F = (0,-\rho^F g_0)^T$, where g_0 is the gravitational acceleration. The velocity profile of the fluid on the rigid boundary is given by:

$$\begin{array}{rcl} g_1(x_1,x_2) &=& \left\{\begin{array}{ll} \left(1-\frac{x_2^2}{H^2}\right)V_0, & (x_1,x_2)\in\Sigma_1\cup\Sigma_3 \\ V_0, & (x_1,x_2)\in\Sigma_2 \end{array}\right. \\ g_1(x_1,x_2) &=& 0, \quad (x_1,x_2)\in\Sigma \end{array}$$

We assume that the density of the structure is constant ρ^S, the cross section is rectangular of thickness h and width ℓ. In this case we have

$$D = \frac{Eh^3\ell}{12}.$$

We assume that the volume forces in the structure have the form

$$f^S(x_1) = -\rho^S g_0\, h + \frac{2\mu V_0}{H^2} x_1, \quad \forall x_1 \in (0, L). \tag{12}$$

Then, we have the following solution for the system (1)–(7):

$$\begin{array}{rcl} u(x_1) & = & 0, \quad \forall x_1 \in (0, L) \\ v_1(x_1, x_2) & = & \left(1 - \frac{x_2^2}{H^2}\right) V_0, \quad \forall (x_1, x_2) \in \Omega_u^F \\ v_2(x_1, x_2) & = & 0, \quad \forall (x_1, x_2) \in \Omega_u^F \\ p(x_1, x_2) & = & \rho^S g_0\, h - \frac{2\mu V_0}{H^2} x_1 + \rho^F g_0 \left(H - x_2\right), \quad \forall (x_1, x_2) \in \Omega_u^F \end{array}$$

and $\alpha = -\rho^S g_0\, h$ and $\beta = \frac{2\mu V_0}{H^2}$ is a solution of (11).

Remark 1 *The term* $\frac{2\mu V_0}{H^2} x_1$ *in (12) is artificial. It was added to obtain a solution where the displacement of the beam is null and the flow is Poisseuille.*

Remark 2 *The uniqueness of the solution is an open problem.*

6 Numerical results

We use the same notations as in the precedent section. The computation has been made in a domain of height $H = 1\ cm$ and length $L = 3\ cm$.

The numerical values of the physical parameters have been token from [4].

The viscosity of the fluid is $\mu = 0.035\ \frac{g}{cm \cdot s}$, its density $\rho^F = 1\ \frac{g}{cm^3}$, $V_0 = 5\ \frac{cm}{s}$.

The thickness of the beam is $h = 0.1\ cm$, the width $\ell = 1\ cm$, the Young modulus $E = 0.75 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, the density $\rho^S = 1.1\ \frac{g}{cm^3}$.

The gravitational acceleration is $g_0 = 981\ \frac{cm}{s^2}$.

The numerical tests have been produced using *freefem++* (see [7]).

The exact solution of the beam equations is given by:

$$u(x_1) = \frac{x_1^2 \left(L - x_1\right)^2}{120D} \left(\left(\frac{2\mu V_0}{H^2} - \beta \right) x_1 + 2L \left(\frac{2\mu V_0}{H^2} - \beta \right) - 5\rho^S g_0\, h - 5\alpha \right).$$

For the fluid we have used the Mixed Finite Element Method, $P2$ Lagrange triangles for the velocity and $P1$ for the pressure.

The cost function has the value $J = 46763$ for $\alpha = 0$ and $\beta = 0$. The minimal value of the cost function $J = 2.32 \cdot 10^{-9}$ was obtained for $\alpha = -\rho^S g_0\, h$ and $\beta = \frac{2\mu V_0}{H^2}$.

The values of the cost function obtained for different values of α and β, are presented in the tables below.

α	β	J
-7.91	0.7	40880.6
-17.91	0.525	32857.6
-27.91	0.35	25738.4
-57.91	0.175	9885.58
-87.91	0	1510.89

α	β	J
-97.91	0.7	441.164
-97.91	0.35	398.015
-97.91	0	357.795
-107.91	0.7	1.46363
-107.91	0.35	2.3e-09

α	β	J
-107.91	0	1.46355
-117.91	0.7	357.776
-117.91	0.35	398.009
-127.91	0.7	1510.81
-137.91	0.35	3586.85

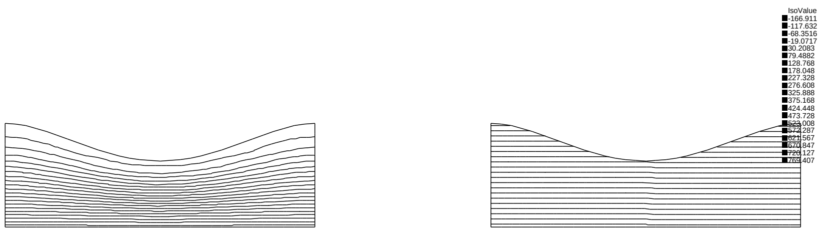

Figure 2: The streamlines (left) and the pression (right) for $\alpha = 0$ and $\beta = 0$

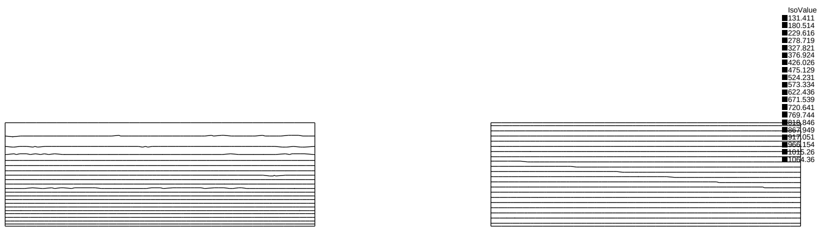

Figure 3: The streamlines (left) and the pression (right) for $\alpha = -\rho^S g_0\, h$ and $\beta = \frac{2\mu V_0}{H^2}$

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