Stable semi-implicit monolithic scheme for interaction between incompressible neo-Hoohean structure and Navier-Stokes fluid

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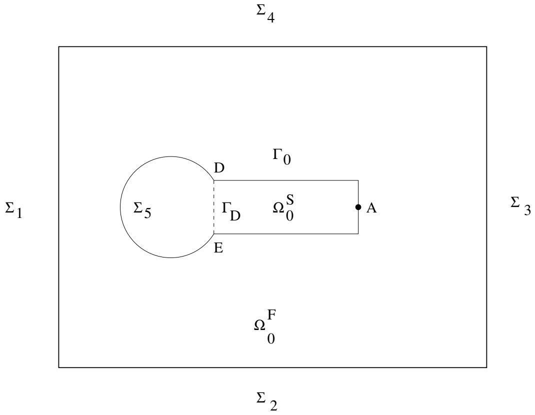
Initial geometrical configuration



Nonlinear elasticity. Notations

$\mathbf{U}^S : \Omega_0^S \times [0, T] \to \mathbb{R}^2$ the displacement of the structure
For $\mathbf{X} \in \Omega_0^S$, $\mathbf{x} = \mathbf{X} + \mathbf{U}^S(\mathbf{X}, t)$ is in Ω_t^S.
$\mathbf{F}(\mathbf{X}, t) = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^S(\mathbf{X}, t)$ the gradient of the deformation
$J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t)$
$\mathbf{\Sigma}(\mathbf{X}, t)$ the second Piola-Kirchhoff stress tensor
$\sigma^S(\mathbf{x}, t)$ the Cauchy stress tensor

$$\sigma^S(\mathbf{x}, t) = \frac{1}{J(\mathbf{X}, t)} \mathbf{F}(\mathbf{X}, t) \mathbf{\Sigma}(\mathbf{X}, t) \mathbf{F}^T(\mathbf{X}, t)$$

The structure is incompressible Neo-Hookean

$$\sigma^S(\mathbf{x}, t) = -p^S(\mathbf{x}, t)\mathbf{I} + \mu^S \left(\mathbf{F}(\mathbf{X}, t) \mathbf{F}^T(\mathbf{X}, t) - \mathbf{I} \right)$$

where p^S is the structure pressure in the Eulerian coordinates, $\mu^S > 0$ is a constant, $\mathbf{I}$ is the unity matrix, $J(\mathbf{X}, t) = 1$.

Nonlinear elasticity equations

$$\begin{aligned} \rho_0^S(\mathbf{X})\frac{\partial^2 \mathbf{U}^S}{\partial t^2}(\mathbf{X},t) - \nabla_{\mathbf{X}}\cdot(\mathbf{F}\boldsymbol{\Sigma})(\mathbf{X},t) &= \rho_0^S(\mathbf{X})\,\mathbf{g}, \quad \text{in } \Omega_0^S \times (0,T) \\ \mathbf{U}^S(\mathbf{X},t) &= 0, \quad \text{on } \Gamma_D \times (0,T) \end{aligned}$$

$\rho_0^S : \Omega_0^S \to \mathbb{R}$ the initial mass density of the structure
$\mathbf{g}$ the acceleration of gravity vector and it is assumed to be constant

Navier-Stokes equations

We denote by $\mathbf{v}^F$ the fluid velocity and by p^F the fluid pressure.

$$\begin{aligned} \rho^F \left(\frac{\partial \mathbf{v}^F}{\partial t} + (\mathbf{v}^F \cdot \nabla)\mathbf{v}^F \right) - 2\mu^F \nabla \cdot \epsilon\left(\mathbf{v}^F\right) + \nabla p^F &= \rho^F \mathbf{g}, \text{ in } \Omega_t^F \\ \nabla \cdot \mathbf{v}^F &= 0, \text{ in } \Omega_t^F \end{aligned}$$

$$\begin{aligned} \sigma^F \mathbf{n}^F &= \mathbf{h}_{in}, \text{ on } \Sigma_1 \\ \sigma^F \mathbf{n}^F &= \mathbf{h}_{out}, \text{ on } \Sigma_3 \\ \mathbf{v}^F &= 0, \text{ on } \Sigma_2 \cup \Sigma_4 \cup \Sigma_5 \end{aligned}$$

$\sigma^F = -p^F \mathbf{I} + 2\mu^S \epsilon\left(\mathbf{v}^F\right)$ the fluid stress tensor

$\epsilon\left(\mathbf{v}^F\right) = \frac{1}{2}\left(\nabla \mathbf{v}^F + \left(\nabla \mathbf{v}^F\right)^T\right)$ the fluid rate of strain tensor

$\mathbf{n}^F$ the unit outer normal vector to $\partial\Omega_t^F$

Interface and initial conditions

$$
\begin{aligned}
\mathbf{v}^F\left(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t\right) &= \frac{\partial \mathbf{U}^S}{\partial t}(\mathbf{X},t),\ \Gamma_0\times(0,T)\\
\left(\sigma^F\mathbf{n}^F\right)_{\left(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t\right)} &= -\left(\mathbf{F}\boldsymbol{\Sigma}\right)(\mathbf{X},t)\,\mathbf{N}^S(\mathbf{X}),\ \Gamma_0\times(0,T)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{U}^S(\mathbf{X},0) &= \mathbf{U}^{S,0}(\mathbf{X}),\ \text{in } \Omega_0^S\\
\frac{\partial \mathbf{U}^S}{\partial t}(\mathbf{X},0) &= \mathbf{V}^{S,0}(\mathbf{X}),\ \text{in } \Omega_0^S\\
\mathbf{v}^F(\mathbf{X},0) &= \mathbf{v}^{F,0}(\mathbf{X}),\ \text{in } \Omega_0^F
\end{aligned}
$$

$\mathbf{N}^S$ the unit outer normal vector to $\partial\Omega_0^S$

Total Lagrangian framework for structure equations

Let $N \in \mathbb{N}^*$ be the number of time steps and $\Delta t = T/N$ the time step. We set $t_n = n\Delta t$ for $n = 0, 1, \dots, N$. Let $\mathbf{V}^{S,n}(\mathbf{X})$ and $\mathbf{U}^{S,n}(\mathbf{X})$ be approximations of $\mathbf{V}^S(\mathbf{X}, t_n)$ and $\mathbf{U}^S(\mathbf{X}, t_n)$. We also use following notations for $n \geq 0$

$$\mathbf{F}^n = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^{S,n},$$
$$\mathbf{\Sigma}^n = -P^{S,n}(\mathbf{F}^n)^{-1}(\mathbf{F}^n)^{-T} + \mu^S\left(\mathbf{I} - (\mathbf{F}^n)^{-1}(\mathbf{F}^n)^{-T}\right).$$

The structure problem will be approached by the implicit Euler scheme

$$\rho_0^S(\mathbf{X})\frac{\mathbf{V}^{S,n+1}(\mathbf{X}) - \mathbf{V}^{S,n}(\mathbf{X})}{\Delta t} - \nabla_{\mathbf{X}} \cdot \left(\mathbf{F}^{n+1}\mathbf{\Sigma}^{n+1}\right)(\mathbf{X}) = \rho_0^S(\mathbf{X})\,\mathbf{g}$$
$$\frac{\mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X})}{\Delta t} = \mathbf{V}^{S,n+1}(\mathbf{X})$$

Weak form in Total Lagrangian framework

$$\mathbf{F}^{n+1} = \mathbf{F}^{n} + \Delta t \nabla_{\mathbf{X}} \mathbf{V}^{S,n+1}$$

Consequently, $\mathbf{F}^{n+1}$ and $\mathbf{\Sigma}^{n+1}$ depend on the velocity $\mathbf{V}^{S,n+1}$ but not in the displacement $\mathbf{U}^{S,n+1}$.
Find $\mathbf{V}^{S,n+1} : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{V}^{S,n+1} = 0$ on Γ_0^D, such that

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S \, d\mathbf{X} + \int_{\Omega_0^S} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}} \mathbf{W}^S \, d\mathbf{X}$$
$$= \int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S \, d\mathbf{X} + \int_{\Gamma_0} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS$$

for all $\mathbf{W}^S : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{W}^S = 0$ on Γ_0^D , subject to

$$\det\left(\mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^{S,n} + \Delta t \nabla_{\mathbf{X}} \mathbf{V}^{S,n+1}\right) = 1, \text{ in } \Omega_0^S.$$

We have assumed that the forces $\mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S$ on the interface Γ_0 are known.

Updated Lagrangian framework. I

$$\mathbf{X} \in \Omega_0^S \to \widehat{\mathbf{x}} \in \widehat{\Omega}^S = \Omega_n^S \to \mathbf{x} \in \Omega_{n+1}^S$$

$$\widehat{\mathbf{x}} = \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X}), \quad \mathbf{x} = \mathbf{X} + \mathbf{U}^{S,n+1}(\mathbf{X})$$

$$\widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = \mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X})$$

Putting $\widehat{\mathbf{F}} = \mathbf{I} + \nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{u}}$, $\widehat{J} = \det\widehat{\mathbf{F}}$ and $J^n = \det\mathbf{F}^n$, we get

$$\mathbf{F}^{n+1}(\mathbf{X}) = \widehat{\mathbf{F}}(\widehat{\mathbf{x}})\,\mathbf{F}^n(\mathbf{X}), \quad J^{n+1}(\mathbf{X}) = \widehat{J}(\widehat{\mathbf{x}})\,J^n(\mathbf{X}).$$

$$\sigma^{S,n+1}(\mathbf{x}) = \left(\frac{1}{J^{n+1}}\mathbf{F}^{\mathbf{n+1}}\mathbf{\Sigma}^{n+1}\left(\mathbf{F}^{n+1}\right)^T\right)(\mathbf{X})$$

For an incompressible material, normally we have $J^n = J^{n+1} = \widehat{J} = 1$.

Updated Lagrangian framework. II

Let us introduce $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$ defined by

$$\widehat{\mathbf{v}}^{S,n+1}(\widehat{\mathbf{x}}) = \mathbf{V}^{S,n+1}(\mathbf{X}), \quad \mathbf{v}^{S,n}(\widehat{\mathbf{x}}) = \mathbf{V}^{S,n}(\mathbf{X}).$$

$$\mathbf{W}^S : \Omega_0^S \to \mathbb{R}^2, \quad \widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2, \quad \mathbf{w}^S : \Omega_{n+1}^S \to \mathbb{R}^2$$

$$\widehat{\mathbf{w}}^S(\widehat{\mathbf{x}}) = \mathbf{w}^S(\mathbf{x}) = \mathbf{W}^S(\mathbf{X}).$$

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S \, d\mathbf{X} = \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$

$$\int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S \, d\mathbf{X} = \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$

$$\int_{\Omega_0^S} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}} \mathbf{W}^S \, d\mathbf{X} = \int_{\Omega_{n+1}^S} \sigma^{S,n+1} : \nabla \mathbf{w}^S \, d\mathbf{x}$$

Updated Lagrangian framework. III

$$\widehat{\boldsymbol{\Sigma}}\left(\widehat{\mathbf{x}}\right)=\widehat{J}(\widehat{\mathbf{x}})\,\widehat{\mathbf{F}}^{-1}\left(\widehat{\mathbf{x}}\right)\sigma^{S,n+1}\left(\mathbf{x}\right)\widehat{\mathbf{F}}^{-T}\left(\widehat{\mathbf{x}}\right),$$

$$\int_{\Omega^S_{n+1}}\sigma^{S,n+1}:\nabla\mathbf{w}^S\,d\mathbf{x}=\int_{\widehat{\Omega}^S}\widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}}:\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{w}}^S\,d\widehat{\mathbf{x}}.$$

But $\widehat{\mathbf{u}}\left(\widehat{\mathbf{x}}\right)=\mathbf{U}^{S,n+1}\left(\mathbf{X}\right)-\mathbf{U}^{S,n}\left(\mathbf{X}\right)=\Delta t\,\widehat{\mathbf{v}}^{S,n+1}\left(\widehat{\mathbf{x}}\right)$ then

$$\widehat{\mathbf{F}}=\mathbf{I}+\Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}.$$

For the incompressible Neo-Hookean materiel, we have

$$\boldsymbol{\Sigma}^{n+1}=-P^{S,n+1}\left(\mathbf{F}^{n+1}\right)^{-1}\left(\mathbf{F}^{n+1}\right)^{-T}+\mu^S\left(\mathbf{I}-\left(\mathbf{F}^{n+1}\right)^{-1}\left(\mathbf{F}^{n+1}\right)^{-T}\right)$$

it follows that

$$\widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} \;=\; -\frac{1}{J^n}\widehat{p}^{S,n+1}\widehat{\mathbf{F}}^{-T}+\frac{\mu^S}{J^n}\left(\widehat{\mathbf{F}}\mathbf{F}^n\left(\mathbf{F}^n\right)^T-\widehat{\mathbf{F}}^{-T}\right)$$

Updated Lagrangian framework. IV

Since $\det \widehat{\mathbf{F}} \approx 1$, we get that $\widehat{\mathbf{F}}^{-T} \approx \mathrm{cof}\left(\widehat{\mathbf{F}}\right)$

$$\widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} \approx -\frac{1}{J^n}\widehat{p}^{S,n+1}\mathbf{I} - \frac{1}{J^n}\widehat{p}^{S,n+1}(\Delta t)\mathrm{cof}\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)$$
$$+\frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \mathbf{I} - (\Delta t)\mathrm{cof}\left(\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\right).$$

We introduce $\widehat{\mathbf{L}}_1\left(\widehat{\mathbf{v}}^{S,n+1}, \widehat{p}^{S,n+1}\right) =$

$$-\frac{1}{J^n}\widehat{p}^{S,n+1}\mathbf{I} + \frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \mathbf{I}\right).$$

Weak form of the Updated Lagrangian framework

Knowing $\mathbf{U}^{S,n} : \Omega_0^S \to \mathbb{R}^2$, $\widehat{\Omega}^S = \Omega_n^S$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$,find $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{S,n+1} = 0$ on Γ_D and $\widehat{p}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}$, such that

$$\int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \widehat{\mathbf{L}}_1 \left(\widehat{\mathbf{v}}^{S,n+1}, \widehat{p}^{S,n+1} \right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$
$$= \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\Gamma_0} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS$$

for all $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2$, $\widehat{\mathbf{w}}^S = 0$ on Γ_D, subject to

$$\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{S,n+1} = 0, \text{ in } \widehat{\Omega}^S.$$

The exact incompressibility condition $\widehat{J} = 1$ gives

$$1 + (\Delta t) \nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{S,n+1} + (\Delta t)^2 \det(\nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}) = 1$$

Weak form: existence and uniqueness

$$\widehat{\mathbb{W}}^S = \left\{ \widehat{\mathbf{w}}^S \in \left(H^1 \left(\widehat{\Omega}^S \right) \right)^2 ; \ \widehat{\mathbf{w}}^S = 0 \text{ on } \Gamma_D \right\}, \qquad \widehat{\mathbb{Q}}^S = L^2 \left(\widehat{\Omega}^S \right)$$

Find $\widehat{\mathbf{v}}^{S,n+1} \in \widehat{\mathbb{W}}^S$, $\widehat{p}^{S,n+1} \in \widehat{\mathbb{Q}}^S$ such that

$$\begin{aligned} \widehat{a}^S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{\mathbf{w}}^S) + \widehat{b}^S(\widehat{\mathbf{w}}^S, \widehat{p}^{S,n+1}) &= \mathcal{L}_S(\widehat{\mathbf{w}}^S), \quad \forall \widehat{\mathbf{w}}^S \in \widehat{\mathbb{W}}^S \\ \widehat{b}^S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{q}^S) &= 0, \quad \forall \widehat{q}^S \in \widehat{\mathbb{Q}}^S \end{aligned}$$

$$\begin{aligned} \widehat{a}^S(\widehat{\mathbf{v}}^{S,n+1}, \widehat{\mathbf{w}}^S) &= \int_{\widehat{\Omega}^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{S,n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} \\ &\quad + \int_{\widehat{\Omega}^S} \frac{\mu^S}{J^n} \left(\Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1} \right) \mathbf{F}^n \left(\mathbf{F}^n \right)^T : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} \\ \widehat{b}^S(\widehat{\mathbf{w}}^S, \widehat{q}^S) &= - \int_{\widehat{\Omega}^S} \frac{1}{J^n} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^S \right) \widehat{q}^S \, d\widehat{\mathbf{x}}. \end{aligned}$$

Proposition The mixed problem has an unique solution.

Stability of the structure problem

Theorem The time advancing scheme for the structure verifies

$$
\begin{aligned}
& \frac{1}{2}\int_{\Omega_0^S} \rho_0^S \left|\mathbf{V}^{S,n+1}\right|^2 d\mathbf{X} + \frac{1}{2}\int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : \mathbf{F}^{n+1} d\mathbf{X} \\
\leq \; & \frac{1}{2}\int_{\Omega_0^S} \rho_0^S \left|\mathbf{V}^{S,n}\right|^2 d\mathbf{X} + \frac{1}{2}\int_{\Omega_0^S} \mu^S \mathbf{F}^{n} : \mathbf{F}^{n} d\mathbf{X} \\
\leq \; & \frac{1}{2}\int_{\Omega_0^S} \rho_0^S \left|\mathbf{V}^{S,0}\right|^2 d\mathbf{X} + \frac{1}{2}\int_{\Omega_0^S} \mu^S \mathbf{F}^{0} : \mathbf{F}^{0} d\mathbf{X}
\end{aligned}
$$

if the right hand side is zero, where $\left|\mathbf{V}^{S,n}\right|^2 = \left(V_1^{S,n}\right)^2 + \left(V_2^{S,n}\right)^2$.

Proof.

$$\widehat{\mathbf{w}}^S = (\Delta t)\widehat{\mathbf{v}}^{S,n+1}$$

$$\begin{aligned}
&\int_{\widehat{\Omega}^S} \widehat{\mathbf{L}}_1\left(\widehat{\mathbf{v}}^{S,n+1}, \widehat{p}^{S,n+1}\right) : (\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\, d\widehat{\mathbf{x}} \\
= &\int_{\widehat{\Omega}^S} \frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\right)\mathbf{F}^n\left(\mathbf{F}^n\right)^T\right) : (\Delta t)\nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}^{S,n+1}\, d\widehat{\mathbf{x}} \\
= &\int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : (\Delta t)\nabla_{\mathbf{X}}\mathbf{V}^{S,n+1} d\mathbf{X} = \int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : \left(\mathbf{F}^{n+1} - \mathbf{F}^n\right) d\mathbf{X}.
\end{aligned}$$

Using $\frac{a^2}{2} - \frac{b^2}{2} \le (a-b)a$, we get

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : \mathbf{F}^{n+1} d\mathbf{X} - \frac{1}{2}\int_{\Omega_0^S} \mu^S \mathbf{F}^n : \mathbf{F}^n d\mathbf{X} \\
\le &\int_{\Omega_0^S} \mu^S \mathbf{F}^{n+1} : \left(\mathbf{F}^{n+1} - \mathbf{F}^n\right) d\mathbf{X}.
\end{aligned}$$

Arbitrary Eulerian Lagrangian (ALE) framework for fluid equations

Ω_n^F, $\mathbf{v}^{F,n}$, $p^{F,n}$
$\widehat{\Omega}^F = \Omega_n^F$, $\mathcal{A}_{n+1} : \Omega_n^F \to \mathbb{R}^2$ by

$$\mathcal{A}_{n+1}(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + \Delta t \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}})$$

We shall construct $\widehat{\boldsymbol{\vartheta}}^{n+1} : \Omega_n^F \to \mathbb{R}^2$ by harmonic extension such that the mesh velocity is zero on the fixed boundary and the mesh velocity is equal to the fluid velocity on the fluid-structure interface.
$\widehat{\mathbf{w}}^F(\widehat{\mathbf{x}}) = \mathbf{w}^F(\mathcal{A}_{n+1}(\widehat{\mathbf{x}}))$, $\mathbf{x} = \mathcal{A}_{n+1}(\widehat{\mathbf{x}})$
The Jacobian of the ALE map is

$$\widehat{J}_{n+1}(\widehat{\mathbf{x}}) = \det(\nabla_{\widehat{\mathbf{x}}}\mathcal{A}_{n+1}(\widehat{\mathbf{x}})) = 1 + \Delta t \nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}}) + (\Delta t)^2 \det(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}})).$$

The time advancing scheme for fluid equations

Find $\widehat{\mathbf{v}}^{F,n+1}$, $\widehat{p}^{F,n+1}$, $\widehat{\boldsymbol{\vartheta}}^{n+1}$, such that

$$
\begin{aligned}
&\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} \\
&+ \int_{\Omega_n^F} \rho^F \left(\left(\left(\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1} \right) \cdot \nabla_{\widehat{\mathbf{x}}} \right) \widehat{\mathbf{v}}^{F,n+1} \right) \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} \\
&+ \frac{(\Delta t)}{2} \int_{\Omega_n^F} \rho^F \det \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} \right) \widehat{\mathbf{v}}^{F,n+1} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} \\
&+ \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{w}}^F \right) d\widehat{\mathbf{x}} \\
&- \int_{\Omega_n^F} \widehat{p}^{F,n+1} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^F) d\widehat{\mathbf{x}} = \mathcal{L}_F(\widehat{\mathbf{w}}^F) + \int_{\Gamma_n} \left(\sigma^F(\widehat{\mathbf{v}}^{F,n+1}, \widehat{p}^{F,n+1}) \mathbf{n}^F \right) \cdot \\
&- \int_{\Omega_n^F} \widehat{q}^F (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{F,n+1}) d\widehat{\mathbf{x}} = 0,
\end{aligned}
$$

$$
\Delta \widehat{\boldsymbol{\vartheta}}^{n+1} = 0 \text{ in } \Omega_n^F, \quad \widehat{\boldsymbol{\vartheta}}^{n+1} = \widehat{\mathbf{v}}^{F,n+1} \text{ on } \Gamma_n, \quad \widehat{\boldsymbol{\vartheta}}^{n+1} = 0 \text{ on } \partial\Omega_n^F \setminus \Gamma_n
$$

Stability of the fluid sheme

$$\mathcal{L}_F(\widehat{\mathbf{w}}^F) = \int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \mathbf{g} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \widehat{\mathbf{w}}^F ds + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot \widehat{\mathbf{w}}^F ds$$

Theorem The time advancing scheme for the fluid verifies

$$\frac{1}{2}\int_{\Omega_{n+1}^F} \rho^F \left|\mathbf{v}^{F,n+1}\right|^2 d\mathbf{x} + (\Delta t)\sum_{k=0}^{n} \int_{\Omega_k^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}) : \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}) d\widehat{\mathbf{x}} \leq \frac{1}{2}\int_{\Omega_0^F} \rho^F \left|\mathbf{v}^{F,0}\right|^2 d\mathbf{X}$$

if g, $\mathbf{h}_{in}$, $\mathbf{h}_{out}$ and the forces acting on Γ_n are zero in the right-hand side and $\int_{\Sigma_1\cup\Sigma_3}(\widehat{\mathbf{v}}^{F,n+1}\cdot\mathbf{n}^F)|\widehat{\mathbf{v}}^{F,n+1}|^2 \geq 0$, where $|\widehat{\mathbf{v}}^{F,n+1}|^2 = (\widehat{\mathbf{v}}_1^{F,n+1})^2 + (\widehat{\mathbf{v}}_2^{F,n+1})^2$.

Proof

$$\widehat{\mathbf{w}}^F = (\Delta t)\widehat{\mathbf{v}}^{F,n+1}$$

$$\begin{aligned}
& \int_{\Omega_n^F} \rho^F \left(\widehat{\mathbf{v}}^{F,n+1} - \mathbf{v}^{F,n}\right) \cdot \widehat{\mathbf{v}}^{F,n+1} d\widehat{\mathbf{x}} \\
+ & \ (\Delta t) \int_{\Omega_n^F} \rho^F \left(\left(\left(\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1} \right) \cdot \nabla_{\widehat{\mathbf{x}}} \right) \widehat{\mathbf{v}}^{F,n+1} \right) \cdot \widehat{\mathbf{v}}^{F,n+1} d\widehat{\mathbf{x}} \\
+ & \ \frac{(\Delta t)^2}{2} \int_{\Omega_n^F} \rho^F \det \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1} \right) \widehat{\mathbf{v}}^{F,n+1} \cdot \widehat{\mathbf{v}}^{F,n+1} d\widehat{\mathbf{x}} \\
+ & \ (\Delta t) \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{F,n+1} \right) d\widehat{\mathbf{x}} = 0.
\end{aligned}$$

By using $[(\widehat{\mathbf{w}} \cdot \nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{v}}] \cdot \widehat{\mathbf{v}} = \frac{1}{2}\widehat{\mathbf{w}} \cdot (\nabla_{\widehat{\mathbf{x}}}|\widehat{\mathbf{v}}|^2)$, we get

$$\begin{aligned}
&\int_{\Omega_n^F} [((\widehat{\mathbf{v}}^{F,n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1}) \cdot \nabla_{\widehat{\mathbf{x}}})\widehat{\mathbf{v}}^{F,n+1}] \cdot \widehat{\mathbf{v}}^{F,n+1} \, d\widehat{\mathbf{x}} \\
= \;& \frac{1}{2}\int_{\Sigma_1 \cup \Sigma_3} \widehat{\mathbf{v}}^{F,n+1} \cdot \mathbf{n}^F |\widehat{\mathbf{v}}^{F,n+1}|^2 d\mathbf{s} + \frac{1}{2}\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1})|\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}}.
\end{aligned}$$

For the last equality, we have used the fact that $\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{F,n+1} = 0$ in Ω_n^F, and the boundary conditions: $\widehat{\mathbf{v}}^{F,n+1} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, $\widehat{\mathbf{v}}^{F,n+1} = \boldsymbol{\vartheta}^{n+1}$ on Γ_n and $\boldsymbol{\vartheta}^{n+1} = 0$ on $\Sigma_1 \cup \Sigma_3$.

$$
\begin{aligned}
& \frac{1}{2}\int_{\Omega_n^F} \rho^F \left|\widehat{\mathbf{v}}^{F,n+1}\right|^2 d\widehat{\mathbf{x}} + \frac{\Delta t}{2}\int_{\Omega_n^F} \rho^F (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\boldsymbol{\vartheta}}^{n+1}) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}} \\
+ \; & \frac{(\Delta t)^2}{2}\int_{\Omega_n^F} \rho^F \det\left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1}\right) |\widehat{\mathbf{v}}^{F,n+1}|^2 d\widehat{\mathbf{x}} \\
= \; & \frac{1}{2}\int_{\Omega_n^F} \rho^F \left|\widehat{\mathbf{v}}^{F,n+1}\right|^2 \widehat{J}_{n+1}\, d\widehat{\mathbf{x}} = \frac{1}{2}\int_{\Omega_{n+1}^F} \rho^F \left|\mathbf{v}^{F,n+1}\right|^2 d\mathbf{x}.
\end{aligned}
$$

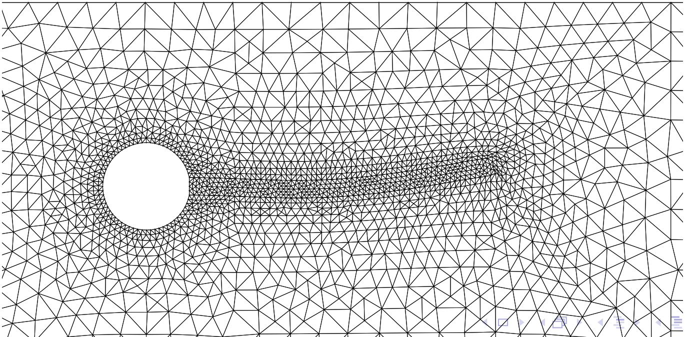
Global moving domain

$$\Omega_n = \Omega_n^F \cup \Gamma_n \cup \Omega_n^S$$

Global velocity and pressure

$$\widehat{\mathbf{v}}^{n+1} : \Omega_n \to \mathbb{R}^2, \quad \widehat{p}^{n+1} : \Omega_n \to \mathbb{R}$$

$$\widehat{\mathbf{v}}^{n+1} = \begin{cases} \widehat{\mathbf{v}}^{F,n+1} \text{ in } \Omega_n^F \\ \widehat{\mathbf{v}}^{S,n+1} \text{ in } \Omega_n^S \end{cases} \quad \widehat{p}^{n+1} = \begin{cases} \widehat{p}^{F,n+1} \text{ in } \Omega_n^F \\ \widehat{p}^{S,n+1} \text{ in } \Omega_n^S \end{cases} \quad \widehat{\mathbf{w}} = \begin{cases} \widehat{\mathbf{w}}^{F} \text{ in } \Omega_n^F \\ \widehat{\mathbf{w}}^{S} \text{ in } \Omega_n^S \end{cases}$$



Monolithic formulation for the fluid-structure equations

Find: the velocity $\widehat{\mathbf{v}}^{n+1} \in \left(H^1\left(\Omega_n\right)\right)^2$, $\widehat{\mathbf{v}}^{n+1} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, the pressure $\widehat{p}^{n+1} \in L^2(\Omega_n)$, the fluid mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1} \in \left(H^1\left(\Omega_n^F\right)\right)^2$, $\widehat{\boldsymbol{\vartheta}}^{n+1} = \widehat{\mathbf{v}}^{F,n+1}$ on Γ_n, $\widehat{\boldsymbol{\vartheta}}^{n+1} = 0$ on $\bigcup_{i=1}^{5} \Sigma_i$

$$\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \left(\left(\left(\widehat{\mathbf{v}}^{n+1} - \widehat{\boldsymbol{\vartheta}}^{n+1}\right) \cdot \nabla_{\widehat{\mathbf{x}}}\right) \widehat{\mathbf{v}}^{n+1}\right) \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}}$$

$$+ \frac{(\Delta t)}{2} \int_{\Omega_n^F} \rho^F \det\left(\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1}\right) \widehat{\mathbf{v}}^{n+1} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{v}}^{n+1}\right) : \epsilon_{\widehat{\mathbf{x}}}\left(\widehat{\mathbf{w}}\right)$$

$$- \int_{\Omega_n} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}\right) \widehat{p}^{n+1} d\widehat{\mathbf{x}}$$

$$+ \int_{\Omega_n^S} \frac{\rho_0^S}{J^n} \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \widehat{\mathbf{L}}_3\left(\widehat{\mathbf{v}}^{n+1}\right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}}$$

$$= \int_{\Omega_n^F} \rho^F \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \mathbf{g} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot$$

$$+ \int_{\Omega_n^S} \frac{\rho_0^S}{J^n} \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \frac{\rho_0^S}{J^n} \mathbf{g} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}},$$

$$-\int_{\Omega_n} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{n+1}) \widehat{q} \, d\widehat{\mathbf{x}} = 0,$$

$$\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \widehat{\vartheta}^{n+1}) \cdot (\nabla_{\widehat{\mathbf{x}}} \widehat{\psi}) d\widehat{\mathbf{x}} = 0$$

for all $\widehat{\mathbf{w}} \in \left(H^1\left(\Omega_n\right)\right)^2$, $\widehat{\mathbf{w}} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, for all $\widehat{q} \in L^2(\Omega_n)$ and for all $\widehat{\psi} \in \left(H_0^1\left(\Omega_n^F\right)\right)^2$, where $\widehat{\mathbf{L}}_3$ is obtained from $\widehat{\mathbf{L}}_1$ by deleting the term with the pressure, i.e.

$$\widehat{\mathbf{L}}_3\left(\widehat{\mathbf{v}}^{n+1}\right) = \frac{\mu^S}{J^n}\left(\left(\mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}\right) \mathbf{F}^n \left(\mathbf{F}^n\right)^T - \mathbf{I}\right).$$

Algorithm 1

We assume that we know Ω_n, Γ_n, $\mathbf{v}^n$.

Step 1: Solve the non-linear system written in Ω_n and get the velocity $\widehat{\mathbf{v}}^{n+1}$, the pressure $\widehat{p}^{n+1}$ and the fluid mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1}$.

Step 2: Define the map $\mathbb{T}_n : \overline{\Omega}_n \to \mathbb{R}^2$ by:

$$\mathbb{T}_n(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + (\Delta t)\widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}})\chi_{\Omega_n^F}(\widehat{\mathbf{x}}) + (\Delta t)\widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}})\chi_{\Omega_n^S}(\widehat{\mathbf{x}})$$

where $\chi_{\Omega_n^F}$ and $\chi_{\Omega_n^S}$ are the characteristic functions of fluid and structure domains.

Step 3: We set $\Omega_{n+1} = \mathbb{T}_n(\Omega_n)$, $\Gamma_{n+1} = \mathbb{T}_n(\Gamma_n)$. We define $\mathbf{v}^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$, $p^{n+1} : \Omega_{n+1} \to \mathbb{R}$ by:

$$\mathbf{v}^{n+1}(\mathbf{x}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}), \;\; p^{n+1}(\mathbf{x}) = \widehat{p}^{n+1}(\widehat{\mathbf{x}}), \;\; \forall \widehat{\mathbf{x}} \in \Omega_n \text{ and } \mathbf{x} = \mathbb{T}_n(\widehat{\mathbf{x}})$$

and $\boldsymbol{\vartheta}^{n+1} : \Omega_{n+1}^F \to \mathbb{R}^2$ by $\boldsymbol{\vartheta}^{n+1}(\mathbf{x}) = \widehat{\boldsymbol{\vartheta}}^{n+1}(\widehat{\mathbf{x}})$. The Lagrangian structure displacement and velocity are defined by

$$\mathbf{U}^{S,n+1}(\mathbf{X}) = \mathbf{U}^{S,n}(\mathbf{X}) + \Delta t\, \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}), \quad \mathbf{V}^{S,n+1}(\mathbf{X}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}})$$

for all $\mathbf{X} \in \Omega_0^S$ and $\widehat{\mathbf{x}} = \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X})$.

Stability of the semi-implicit monolithic scheme

Theorem. The **Algorithm 1** verifies

$$
\begin{aligned}
& \frac{1}{2}\int_{\Omega^F_{n+1}} \rho^F \left|\mathbf{v}^{F,n+1}\right|^2 d\mathbf{x} \\
& +(\Delta t)\sum_{k=0}^{n}\int_{\Omega^F_k} 2\mu^F \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}) : \epsilon_{\widehat{\mathbf{x}}}(\widehat{\mathbf{v}}^{F,k+1}) d\widehat{\mathbf{x}} \\
& +\frac{1}{2}\int_{\Omega^S_0} \rho^S_0 \left|\mathbf{V}^{S,n+1}\right|^2 d\mathbf{X} + \frac{1}{2}\int_{\Omega^S_0} \mu^S \mathbf{F}^{n+1} : \mathbf{F}^{n+1} d\mathbf{X} \\
\leq \;& \frac{1}{2}\int_{\Omega^F_0} \rho^F \left|\mathbf{v}^{F,0}\right|^2 d\mathbf{X} + \frac{1}{2}\int_{\Omega^S_0} \rho^S_0 \left|\mathbf{V}^{S,0}\right|^2 d\mathbf{X} + \frac{1}{2}\int_{\Omega^S_0} \mu^S \mathbf{F}^0 : \mathbf{F}^0 d\mathbf{X}
\end{aligned}
$$

if g, $\mathbf{h}_{in}$, $\mathbf{h}_{out}$ are zero and $\int_{\Sigma_1\cup\Sigma_3}(\widehat{\mathbf{v}}^{F,n+1}\cdot\mathbf{n}^F)|\widehat{\mathbf{v}}^{F,n+1}|^2 \geq 0$.

Algorithm 2

Step 1-1: Find the velocity $\widehat{\mathbf{v}}^{n+1} \in \left(H^1(\Omega_n)\right)^2$, $\widehat{\mathbf{v}}^{n+1} = 0$ on $\Sigma_2 \cup \Sigma_4 \cup \Sigma_5$, the pressure $\widehat{p}^{F,n+1} \in L^2(\Omega_n^F)$, such that:

$$
\begin{aligned}
&\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \left(\left((\mathbf{v}^n - \vartheta^n) \cdot \nabla_{\widehat{\mathbf{x}}} \right) \widehat{\mathbf{v}}^{n+1} \right) \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} \\
&+ \int_{\Omega_n^F} 2\mu^F \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{v}}^{n+1} \right) : \epsilon_{\widehat{\mathbf{x}}} \left(\widehat{\mathbf{w}} \right) d\widehat{\mathbf{x}} \\
&- \int_{\Omega_n^F} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}} \right) \widehat{p}^{F,n+1} d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \frac{1}{\varepsilon} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{n+1} \right) \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}} \right) d\widehat{\mathbf{x}} \\
&+ \int_{\Omega_n^S} \rho_0^S \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \widehat{\mathbf{L}}_4 \left(\widehat{\mathbf{v}}^{n+1} \right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} \\
&= \int_{\Omega_n^F} \rho^F \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \mathbf{g} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot \\
&+ \int_{\Omega_n^S} \rho_0^S \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \rho_0^S \mathbf{g} \cdot \widehat{\mathbf{w}}\, d\widehat{\mathbf{x}}, \\
&- \int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{n+1}) \widehat{q}^F\, d\widehat{\mathbf{x}} = 0,
\end{aligned}
$$

where

$$\widehat{\mathbf{L}}_4\left(\widehat{\mathbf{v}}^{n+1}\right) = \mu^S\left(\left(\mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}\right) \mathbf{F}^n \left(\mathbf{F}^n\right)^T - \mathrm{cof}\left(\mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}\right)\right)$$

Step 1-2: Find the fluid mesh velocity $\widehat{\boldsymbol{\vartheta}}^{n+1} \in \left(H^1\left(\Omega_n^F\right)\right)^2$, $\widehat{\boldsymbol{\vartheta}}^{n+1} = \widehat{\mathbf{v}}^{F,n+1}$ on Γ_n, $\widehat{\boldsymbol{\vartheta}}^{n+1} = 0$ on $\bigcup_{i=1}^5 \Sigma_i$ such that

$$\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\vartheta}}^{n+1}) \cdot (\nabla_{\widehat{\mathbf{x}}} \widehat{\boldsymbol{\psi}}) d\widehat{\mathbf{x}} = 0$$

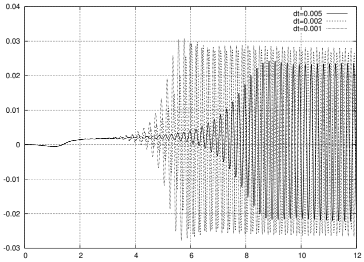
for all $\widehat{\boldsymbol{\psi}} \in \left(H_0^1\left(\Omega_n^F\right)\right)^2$.
The **Steps 2, 3** are the same as in **Algorithm 1**.

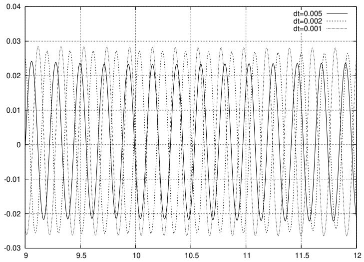
Numerical test

The numerical tests have been produced using *FreeFem++*. We have tested the benchmark FSI3 from Turek 2006.
The fluid: length $L = 2.5\ m$, width $H = 0.41\ m$, dynamic viscosity $\mu^F = 1\ Kg/(m\,s)$, mass density $\rho^F = 1000 Kg/m^3$.
The flexible structure: length $\ell = 0.35\ m$, thickness $h = 0.02\ m$, mass density $\rho^S = 1000 Kg/m^3$ and $\mu^S = 2 \times 10^6\ Kg/(m\,s^2)$. It is attached to the fixed cylinder of center $(x_C, y_C) = (0.2, 0.2)\ m$ and radius $r = 0.5\ m$.
The boundary condition at the inflow Σ_1 is $\mathbf{v} = \mathbf{v}_{in}$, with

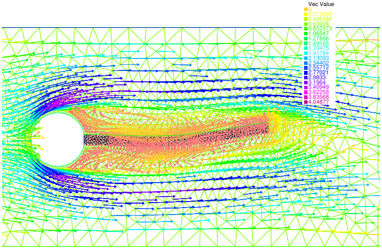
$$\mathbf{v}_{in}(x_1, x_2, t) = \begin{cases} \left(1.5\,\overline{U}\frac{x_2(H-x_2)}{(H/2)^2}\frac{(1-\cos(\pi t/2))}{2},\ 0\right), & 0 \le t \le 2 \\ \left(1.5\,\overline{U}\frac{x_2(H-x_2)}{(H/2)^2},\ 0\right), & 2 \le t \le T = 12 \end{cases}$$

and $\overline{U} = 1.8$.





For $\Delta t = 0.005$, the amplitude is 0.022 m, the frequency 4.72 Hz,
for $\Delta t = 0.002$, the amplitude is 0.025 m, the frequency 4.77 Hz,
for $\Delta t = 0.001$, the amplitude is 0.027 m, the frequency 5 Hz.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2.77021
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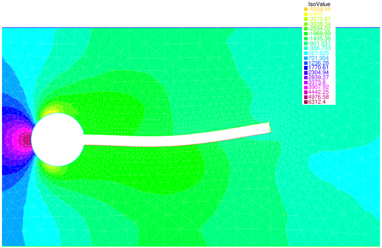
4.04877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701.954
1236.28
1770.61
2304.94
2839.27
3373.6
3907.92
4442.25
4976.58
6312.4

Thank you!