Updated Lagrangian/Arbitrary Lagrangian Eulerian framework for interaction between a compressible Neo-Hookean structure and an incompressible fluid

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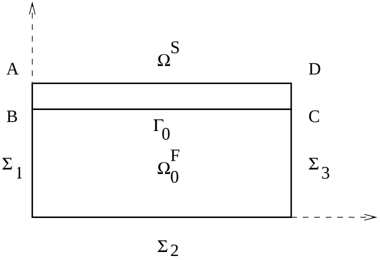
Initial (left) and intermediate (right) geometrical configuration



$$\partial\Omega_0^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_0, \qquad \partial\Omega_t^F = \Sigma_1 \cup \Sigma_2 \cup \Sigma_3 \cup \Gamma_t.$$

Nonlinear elasticity. Notations

$\mathbf{U}^S : \Omega_0^S \times [0, T] \to \mathbb{R}^2$ the displacement of the structure
For $\mathbf{X} \in \Omega_0^S$, $\mathbf{x} = \mathbf{X} + \mathbf{U}^S(\mathbf{X}, t)$ is in Ω_t^S.
$\mathbf{F}(\mathbf{X}, t) = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^S(\mathbf{X}, t)$ the gradient of the deformation
$J(\mathbf{X}, t) = \det \mathbf{F}(\mathbf{X}, t)$
$\mathbf{\Sigma}$ the second Piola-Kirchhoff stress tensor

The structure is homogeneous, isotropic and it can be described by the compressible Neo-Hookean constitutive equation

$$\mathbf{\Sigma} = \lambda^S (\ln J) \mathbf{F}^{-1} \mathbf{F}^{-T} + \mu^S \left(\mathbf{I} - \mathbf{F}^{-1} \mathbf{F}^{-T} \right)$$

where λ^S, μ^S are the Lamé constants of the linearized theory.
Simo & Pister 1984

Nonlinear elasticity equations

$$
\begin{aligned}
\rho_0^S(\mathbf{X})\frac{\partial^2 \mathbf{U}^S}{\partial t^2}(\mathbf{X},t) - \nabla_{\mathbf{X}}\cdot(\mathbf{F\Sigma})(\mathbf{X},t) &= \rho_0^S(\mathbf{X})\,\mathbf{g},\ \Omega_0^S \times (0,T) \\
\mathbf{U}^S(\mathbf{X},t) &= 0, \quad \text{on } \Gamma_0^D \times (0,T) \\
(\mathbf{F\Sigma})(\mathbf{X},t)\,\mathbf{N}^S(\mathbf{X}) &= 0, \quad \text{on } \Gamma_0^N \times (0,T)
\end{aligned}
$$

$$\Gamma_0^D = [AB] \cup [CD], \quad \Gamma_0^N = [DA]$$

$\rho_0^S : \Omega_0^S \to \mathbb{R}$ the initial mass density of the structure
$\mathbf{g}$ the acceleration of gravity vector and it is assumed to be constant
$\mathbf{N}^S$ is the unit outer normal vector along the boundary $\partial\Omega_0^S$

Navier-Stokes equations

We denote by $\mathbf{v}^F$ the fluid velocity and by p^F the fluid pressure.

$$\begin{aligned}
\rho^F\left(\frac{\partial \mathbf{v}^F}{\partial t} + (\mathbf{v}^F\cdot\nabla)\mathbf{v}^F\right) - 2\mu^F\nabla\cdot\epsilon\left(\mathbf{v}^F\right) + \nabla p^F &= \rho^F\mathbf{g},\ \text{in } \Omega_t^F \\
\nabla\cdot\mathbf{v}^F &= 0,\ \text{in } \Omega_t^F \\
\sigma^F\mathbf{n}^F &= \mathbf{h}_{in},\ \text{on } \Sigma_1 \\
\sigma^F\mathbf{n}^F &= \mathbf{h}_{out},\ \text{on } \Sigma_3 \\
\mathbf{v}^F &= 0,\ \text{on } \Sigma_2
\end{aligned}$$

$\sigma^F = -p^F\mathbf{I} + 2\mu^S\epsilon\left(\mathbf{v}^F\right)$ the fluid stress tensor

$\epsilon\left(\mathbf{v}^F\right) = \frac{1}{2}\left(\nabla\mathbf{v}^F + \left(\nabla\mathbf{v}^F\right)^T\right)$ the fluid rate of strain tensor

Interface and initial conditions

$$\mathbf{v}^F\left(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t\right) = \frac{\partial \mathbf{U}^S}{\partial t}(\mathbf{X},t),\ \Gamma_0\times(0,T)$$

$$\left(\sigma^F\mathbf{n}^F\right)_{\left(\mathbf{X}+\mathbf{U}^S(\mathbf{X},t),t\right)}\omega(\mathbf{X},t) = -(\mathbf{F}\boldsymbol{\Sigma})(\mathbf{X},t)\,\mathbf{N}^S(\mathbf{X}),\ \Gamma_0\times(0,T)$$

where $\omega(\mathbf{X},t)=\left\|\mathrm{cof}(\mathbf{F})\,\mathbf{N}^S\right\|_{\mathbb{R}^2}=\left\|J\mathbf{F}^{-T}\mathbf{N}^S\right\|_{\mathbb{R}^2}$

$$\int_{\Gamma_t}\left(\sigma^F\mathbf{n}^F\right)_{(s,t)}ds=\int_{\Gamma_0}\left(\sigma^F\mathbf{n}^F\right)_{\left(S+\mathbf{U}^S(S,t),t\right)}\omega(S,t)\,dS$$

$$\mathbf{U}^S(\mathbf{X},0) = \mathbf{U}^{S,0}(\mathbf{X}),\ \text{in } \Omega_0^S$$

$$\frac{\partial \mathbf{U}^S}{\partial t}(\mathbf{X},0) = \mathbf{V}^{S,0}(\mathbf{X}),\ \text{in } \Omega_0^S$$

$$\mathbf{v}^F(\mathbf{X},0) = \mathbf{v}^{F,0}(\mathbf{X}),\ \text{in } \Omega_0^F$$

Total Lagrangian framework for structure equations

Let $N \in \mathbb{N}^*$ be the number of time steps and $\Delta t = T/N$ the time step. We set $t_n = n\Delta t$ for $n = 0, 1, \ldots, N$. Let $\mathbf{V}^{S,n}(\mathbf{X})$ and $\mathbf{U}^{S,n}(\mathbf{X})$ be approximations of $\mathbf{V}^S(\mathbf{X}, t_n)$ and $\mathbf{U}^S(\mathbf{X}, t_n)$. We also use following notations for $n \geq 0$

$$\mathbf{F}^n = \mathbf{I} + \nabla_{\mathbf{X}} \mathbf{U}^{S,n},$$

$$\mathbf{\Sigma}^n = \lambda^S (\ln J^n) (\mathbf{F}^n)^{-1} (\mathbf{F}^n)^{-T} + \mu^S \left(\mathbf{I} - (\mathbf{F}^n)^{-1} (\mathbf{F}^n)^{-T} \right).$$

The structure problem will be approached by the implicit Euler scheme

$$\rho_0^S(\mathbf{X}) \frac{\mathbf{V}^{S,n+1}(\mathbf{X}) - \mathbf{V}^{S,n}(\mathbf{X})}{\Delta t} - \nabla_{\mathbf{X}} \cdot \left(\mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \right)(\mathbf{X}) = \rho_0^S(\mathbf{X})\, \mathbf{g}$$

$$\frac{\mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X})}{\Delta t} = \mathbf{V}^{S,n+1}(\mathbf{X})$$

Weak form in Total Lagrangian framework

$$\mathbf{F}^{n+1} = \mathbf{F}^n + \Delta t \nabla_{\mathbf{X}} \mathbf{V}^{S,n+1}$$

Consequently, $\mathbf{F}^{n+1}$ and $\mathbf{\Sigma}^{n+1}$ depend on the velocity $\mathbf{V}^{S,n+1}$ but not in the displacement $\mathbf{U}^{S,n+1}$.
Find $\mathbf{V}^{S,n+1} : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{V}^{S,n+1} = 0$ on Γ_0^D, such that

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S \, d\mathbf{X} + \int_{\Omega_0^S} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}} \mathbf{W}^S \, d\mathbf{X}$$
$$= \int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S \, d\mathbf{X} + \int_{\Gamma_0} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS$$

for all $\mathbf{W}^S : \Omega_0^S \to \mathbb{R}^2$, $\mathbf{W}^S = 0$ on Γ_0^D. For instant, we have assumed that the forces $\mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} \mathbf{N}^S$ on the interface Γ_0 are known.

Updated Lagrangian framework. I

$$\mathbf{X} \in \Omega_0^S \to \widehat{\mathbf{x}} \in \widehat{\Omega}^S = \Omega_n^S \to \mathbf{x} \in \Omega_{n+1}^S$$

$$\widehat{\mathbf{x}} = \mathbf{X} + \mathbf{U}^{S,n}(\mathbf{X}), \quad \mathbf{x} = \mathbf{X} + \mathbf{U}^{S,n+1}(\mathbf{X})$$

$$\widehat{\mathbf{u}}(\widehat{\mathbf{x}}) = \mathbf{U}^{S,n+1}(\mathbf{X}) - \mathbf{U}^{S,n}(\mathbf{X})$$

Putting $\widehat{\mathbf{F}} = \mathbf{I} + \nabla_{\widehat{\mathbf{x}}}\widehat{\mathbf{u}}$, $\widehat{J} = \det \widehat{\mathbf{F}}$ and $J^n = \det \mathbf{F}^n$, we get

$$\mathbf{F}^{n+1}(\mathbf{X}) = \widehat{\mathbf{F}}(\widehat{\mathbf{x}})\,\mathbf{F}^n(\mathbf{X}), \quad J^{n+1}(\mathbf{X}) = \widehat{J}(\widehat{\mathbf{x}})\,J^n(\mathbf{X}).$$

$$\sigma^{S,n+1}(\mathbf{x}) = \left(\frac{1}{J^{n+1}}\mathbf{F}^{\mathbf{n+1}}\boldsymbol{\Sigma}^{n+1}\left(\mathbf{F}^{n+1}\right)^T\right)(\mathbf{X}), \quad \rho^{S,n}(\widehat{\mathbf{x}}) = \frac{\rho_0^S(\mathbf{X})}{J^n(\mathbf{X})}$$

Updated Lagrangian framework. II

Let us introduce $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \to \mathbb{R}^2$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \to \mathbb{R}^2$ defined by

$$\widehat{\mathbf{v}}^{S,n+1}\left(\widehat{\mathbf{x}}\right) = \mathbf{V}^{S,n+1}\left(\mathbf{X}\right), \quad \mathbf{v}^{S,n}\left(\widehat{\mathbf{x}}\right) = \mathbf{V}^{S,n}\left(\mathbf{X}\right).$$

$$\mathbf{W}^S : \Omega_0^S \to \mathbb{R}^2, \quad \widehat{\mathbf{w}}^S : \widehat{\Omega}^S \to \mathbb{R}^2, \quad \mathbf{w}^S : \Omega_{n+1}^S \to \mathbb{R}^2$$

$$\widehat{\mathbf{w}}^S\left(\widehat{\mathbf{x}}\right) = \mathbf{w}^S\left(\mathbf{x}\right) = \mathbf{W}^S\left(\mathbf{X}\right).$$

$$\int_{\Omega_0^S} \rho_0^S \frac{\mathbf{V}^{S,n+1} - \mathbf{V}^{S,n}}{\Delta t} \cdot \mathbf{W}^S \, d\mathbf{X} = \int_{\widehat{\Omega}^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$

$$\int_{\Omega_0^S} \rho_0^S \mathbf{g} \cdot \mathbf{W}^S \, d\mathbf{X} = \int_{\widehat{\Omega}^S} \rho^{S,n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}$$

$$\int_{\Omega_0^S} \mathbf{F}^{n+1} \mathbf{\Sigma}^{n+1} : \nabla_{\mathbf{X}} \mathbf{W}^S \, d\mathbf{X} = \int_{\Omega_{n+1}^S} \sigma^{S,n+1} : \nabla \mathbf{w}^S \, d\mathbf{x}$$

Updated Lagrangian framework. III

Let us introduce the tensor

$$\widehat{\boldsymbol{\Sigma}}\left(\widehat{\mathbf{x}}\right) = \widehat{J}\left(\widehat{\mathbf{x}}\right)\widehat{\mathbf{F}}^{-1}\left(\widehat{\mathbf{x}}\right)\sigma^{S,n+1}\left(\mathbf{x}\right)\widehat{\mathbf{F}}^{-T}\left(\widehat{\mathbf{x}}\right),$$

we get

$$\int_{\Omega^S_{n+1}} \sigma^{S,n+1} : \nabla \mathbf{w}^S \, d\mathbf{x} = \int_{\widehat{\Omega}^S} \widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}} : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}}.$$

But $\widehat{\mathbf{u}}\left(\widehat{\mathbf{x}}\right) = \mathbf{U}^{S,n+1}\left(\mathbf{X}\right) - \mathbf{U}^{S,n}\left(\mathbf{X}\right) = \Delta t\, \widehat{\mathbf{v}}^{S,n+1}\left(\widehat{\mathbf{x}}\right)$ then

$$\widehat{\mathbf{F}} = \mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}.$$

For the compressible Neo-Hookean materiel, we have

$$\sigma^{S,n+1} = \frac{\lambda^S}{J^{n+1}} (\ln J^{n+1})\mathbf{I} + \frac{\mu^S}{J^{n+1}} \left(\mathbf{F}^{n+1}\left(\mathbf{F}^{n+1}\right)^T - \mathbf{I}\right)$$

it follows that

$$\widehat{\boldsymbol{\Sigma}} = \frac{\lambda^S}{J^n}(\ln J^n + \ln \widehat{J})\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T} + \frac{\mu^S}{J^n}\left(\mathbf{F}^n\left(\mathbf{F}^n\right)^T - \widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T}\right).$$

$\widehat{\mathbf{F}}$ depends on $\widehat{\mathbf{v}}^{S,n+1}$ and $\widehat{\boldsymbol{\Sigma}}$ depends on $\widehat{\mathbf{v}}^{S,n+1}$ and $\mathbf{F}^n\left(\mathbf{X}\right)$.

Weak form of the Updated Lagrangian framework

$$\det(\mathbf{I}+\mathbf{A}) \approx 1 + tr(\mathbf{A}), \quad (\mathbf{I}+\mathbf{A})^{-1} \approx \mathbf{I}-\mathbf{A}, \quad \ln(1+x) \approx x$$

We linearize the map $\widehat{\mathbf{v}}^{S,n+1} \longrightarrow \widehat{\mathbf{F}}\widehat{\boldsymbol{\Sigma}}$ by $\widehat{\mathbf{L}}\left(\widehat{\mathbf{v}}^{S,n+1}\right)$

$$\begin{aligned} = & \ \frac{\lambda^S}{J^n} \ln J^n \left(\mathbf{I} - \Delta t \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1} \right)^T \right) + \frac{\lambda^S}{J^n} (\Delta t) tr(\nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1}) \mathbf{I} \\ + & \ \frac{\mu^S}{J^n} \left(\left(\mathbf{I} + \Delta t \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1} \right) \mathbf{F}^n (\mathbf{F}^n)^T - \mathbf{I} + \Delta t \left(\nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{v}}^{S,n+1} \right)^T \right). \end{aligned}$$

Knowing $\mathbf{U}^{S,n} : \Omega_0^S \rightarrow \mathbb{R}^2$, $\widehat{\Omega}^S = \Omega_n^S$ and $\mathbf{v}^{S,n} : \widehat{\Omega}^S \rightarrow \mathbb{R}^2$, find $\widehat{\mathbf{v}}^{S,n+1} : \widehat{\Omega}^S \rightarrow \mathbb{R}^2$, $\widehat{\mathbf{v}}^{S,n+1} = 0$ on Γ_0^D such that

$$\begin{aligned} \int_{\widehat{\Omega}^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{S,n+1} - \mathbf{v}^{S,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\widehat{\Omega}^S} \widehat{\mathbf{L}}\left(\widehat{\mathbf{v}}^{S,n+1}\right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} \\ = \int_{\widehat{\Omega}^S} \rho^{S,n} \mathbf{g} \cdot \widehat{\mathbf{w}}^S \, d\widehat{\mathbf{x}} + \int_{\Gamma_0} \mathbf{F}^{n+1} \boldsymbol{\Sigma}^{n+1} \mathbf{N}^S \cdot \mathbf{W}^S \, dS \end{aligned}$$

for all $\widehat{\mathbf{w}}^S : \widehat{\Omega}^S \rightarrow \mathbb{R}^2$, $\widehat{\mathbf{w}}^S = 0$ on Γ_0^D.

Arbitrary Lagrangian Eulerian (ALE) framework. Notations

The reference fluid domain $\widehat{\Omega}^F = \Omega_n^F$, the interface Γ_n
The velocity of the fluid mesh $\boldsymbol{\vartheta}^n = (\vartheta_1^n, \vartheta_2^n)^T$ is the solution of

$$\Delta \boldsymbol{\vartheta}^n = 0 \text{ in } \Omega_n^F, \quad \boldsymbol{\vartheta}^n = 0 \text{ on } \partial\Omega_n^F \setminus \Gamma_n, \quad \boldsymbol{\vartheta}^n = \mathbf{v}^{F,n} \text{ on } \Gamma_n$$

The ALE map $\mathcal{A}_{t_{n+1}} : \overline{\Omega}_n^F \to \mathbb{R}^2$

$$\mathcal{A}_{t_{n+1}}(\widehat{x}_1, \widehat{x}_2) = (\widehat{x}_1 + \Delta t \vartheta_1^n, \widehat{x}_2 + \Delta t \vartheta_2^n).$$

We define $\Omega_{n+1}^F = \mathcal{A}_{t_{n+1}}(\Omega_n^F)$ and $\Gamma_{n+1} = \mathcal{A}_{t_{n+1}}(\Gamma_n)$
We introduce $\widehat{\mathbf{v}}^{F,n+1} : \Omega_n^F \to \mathbb{R}^2$ and $\widehat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$ defined by

$$\widehat{\mathbf{v}}^{F,n+1}(\widehat{\mathbf{x}}) = \mathbf{v}^{F,n+1}(\mathbf{x}), \quad \widehat{p}^{F,n+1}(\widehat{\mathbf{x}}) = p^{F,n+1}(\mathbf{x}),$$

$\forall \widehat{\mathbf{x}} \in \Omega_n^F, \ \mathbf{x} = \mathcal{A}_{t_{n+1}}(\widehat{\mathbf{x}}) \in \Omega_{n+1}^F$

Time discretization of the fluid equations. Weak form

Find $\widehat{\mathbf{v}}^{F,n+1} : \Omega_n^F \to \mathbb{R}^2$ such that $\widehat{\mathbf{v}}^{F,n+1} = 0$ on Σ_2 and $\widehat{p}^{F,n+1} : \Omega_n^F \to \mathbb{R}$ such that:

$$\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n+1}}{\Delta t} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \left(\left(\left(\mathbf{v}^{F,n} - \vartheta^n \right) \cdot \nabla_{\widehat{\mathbf{x}}} \right) \widehat{\mathbf{v}}^{F,n+1} \right) \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}}$$

$$- \int_{\Omega_n^F} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}}^F \right) \widehat{p}^{F,n+1} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} 2\mu^F \epsilon \left(\widehat{\mathbf{v}}^{F,n+1} \right) : \epsilon \left(\widehat{\mathbf{w}}^F \right) d\widehat{\mathbf{x}}$$

$$= \mathcal{L}_F(\widehat{\mathbf{w}}^F) + \int_{\Gamma_n} \left(\sigma^F(\widehat{\mathbf{v}}^{F,n+1}, \widehat{p}^{F,n+1}) \mathbf{n}^F \right) \cdot \widehat{\mathbf{w}}^F ds,$$

$$\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{F,n+1}) \widehat{q} \, d\widehat{\mathbf{x}} = 0,$$

for all $\widehat{\mathbf{w}}^F : \Omega_n^F \to \mathbb{R}^2$ such that $\widehat{\mathbf{w}}^F = 0$ on Σ_2 and for all $\widehat{q} : \Omega_n^F \to \mathbb{R}$, where $\mathcal{L}_F(\widehat{\mathbf{w}}^F)$ is

$$\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{F,n}}{\Delta t} \cdot \widehat{\mathbf{w}}^F d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \mathbf{g} \cdot \widehat{\mathbf{w}}^F + \int_{\Sigma_1} \mathbf{h}_{in}^{n+1} \cdot \widehat{\mathbf{w}}^F + \int_{\Sigma_3} \mathbf{h}_{out}^{n+1} \cdot \widehat{\mathbf{w}}^F.$$

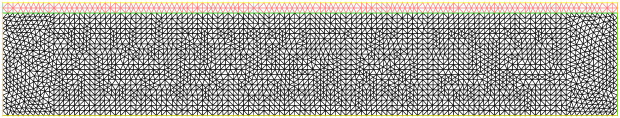
Global moving domain

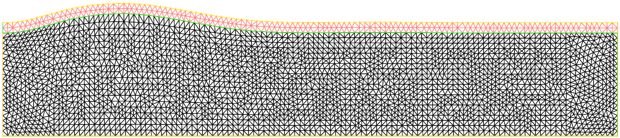
$$\Omega_n = \Omega_n^F \cup \Omega_n^S$$

Global velocity and pressure

$$\widehat{\mathbf{v}}^{n+1} : \Omega_n \to \mathbb{R}^2, \quad \widehat{p}^{n+1} : \Omega_n \to \mathbb{R}$$

$$\widehat{\mathbf{v}}^{n+1} = \begin{cases} \widehat{\mathbf{v}}^{F,n+1} & \text{in } \Omega_n^F \\ \widehat{\mathbf{v}}^{S,n+1} & \text{in } \Omega_n^S \end{cases} \quad \widehat{p}^{n+1} = \begin{cases} \widehat{p}^{F,n+1} & \text{in } \Omega_n^F \\ \widehat{p}^{S,n+1} & \text{in } \Omega_n^S \end{cases} \quad \widehat{\mathbf{w}} = \begin{cases} \widehat{\mathbf{w}}^F & \text{in } \Omega_n^F \\ \widehat{\mathbf{w}}^S & \text{in } \Omega_n^S \end{cases}$$





Monolithic formulation for the fluid-structure equations

Find $\widehat{\mathbf{v}}^{n+1} : \Omega_n \to \mathbb{R}^2$, $\widehat{\mathbf{v}}^{n+1} = 0$ on $\Sigma_2 \cup \Gamma_0^D$ and $\widehat{p}^{n+1} : \Omega_n \to \mathbb{R}$, $\widehat{p}^{n+1} = 0$ in Ω_n^S, such that:

$$\int_{\Omega_n^F} \rho^F \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} \rho^F \left(\left((\mathbf{v}^n - \boldsymbol{\vartheta}^n) \cdot \nabla_{\widehat{\mathbf{x}}} \right) \widehat{\mathbf{v}}^{n+1} \right) \cdot \widehat{\mathbf{w}} d\widehat{\mathbf{x}}$$
$$- \int_{\Omega_n^F} \left(\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{w}} \right) \widehat{p}^{n+1} d\widehat{\mathbf{x}} + \int_{\Omega_n^F} 2\mu^F \epsilon \left(\widehat{\mathbf{v}}^{n+1} \right) : \epsilon \left(\widehat{\mathbf{w}} \right) d\widehat{\mathbf{x}}$$
$$+ \int_{\Omega_n^S} \rho^{S,n} \frac{\widehat{\mathbf{v}}^{n+1}}{\Delta t} \cdot \widehat{\mathbf{w}} \, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \widehat{\mathbf{L}} \left(\widehat{\mathbf{v}}^{n+1} \right) : \nabla_{\widehat{\mathbf{x}}} \widehat{\mathbf{w}} \, d\widehat{\mathbf{x}}$$
$$= \mathcal{L}_F(\widehat{\mathbf{w}}) + \int_{\Omega_n^S} \rho^{S,n} \frac{\mathbf{v}^n}{\Delta t} \cdot \widehat{\mathbf{w}} \, d\widehat{\mathbf{x}} + \int_{\Omega_n^S} \rho^{S,n} \mathbf{g} \cdot \widehat{\mathbf{w}} \, d\widehat{\mathbf{x}},$$
$$\int_{\Omega_n^F} (\nabla_{\widehat{\mathbf{x}}} \cdot \widehat{\mathbf{v}}^{n+1}) \widehat{q} \, d\widehat{\mathbf{x}} = 0,$$

for all $\widehat{\mathbf{w}} : \Omega_n \to \mathbb{R}^2$, $\widehat{\mathbf{w}} = 0$ on $\Sigma_2 \cup \Gamma_0^D$ and for all $\widehat{q} : \Omega_n \to \mathbb{R}$.

Finite element discretization

Triangular $\mathbb{P}_1 +$ *bubble* for the velocity and $\mathbb{P}_1$ for the pressure.
The velocity, the pressure as well as the test functions are continuous all over the global domain Ω_n.
If the solution of the monolithic is sufficiently smooth, the continuity of stress at the interface holds in a weak sense.

We have added the term $\epsilon \int_{\Omega_n} \widehat{p}^{n+1}\widehat{q}$, then the bellow system has an unique solution and $p^S = 0$ on Ω_n^S.

$$\begin{bmatrix} A & B^T & 0 \\ B & \epsilon M^F & 0 \\ 0 & 0 & \epsilon M^S \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ p^F \\ p^S \end{bmatrix} = \begin{bmatrix} \mathcal{L} \\ 0 \\ 0 \end{bmatrix}$$

We have used the LU algorithm for solving the linear system.

Time advancing schema from n to $n+1$

We assume that we know Ω_n, $\mathbf{v}^n$, p^n.
Step 1: Compute $\boldsymbol{\vartheta}^n$
Step 2: Solve the linear system and get the velocity $\widehat{\mathbf{v}}^{n+1}$ and the pressure $\widehat{p}^{n+1}$
Step 3: We define the map $\mathbb{T}_n : \overline{\Omega}_n \to \mathbb{R}^2$ by:

$$\mathbb{T}_n(\widehat{\mathbf{x}}) = \widehat{\mathbf{x}} + (\Delta t)\boldsymbol{\vartheta}^n(\widehat{\mathbf{x}})\chi_{\Omega_n^F}(\widehat{\mathbf{x}}) + (\Delta t)\mathbf{v}^n(\widehat{\mathbf{x}})\chi_{\Omega_n^S}(\widehat{\mathbf{x}})$$

Step 4: We set $\Omega_{n+1} = \mathbb{T}_n(\Omega_n)$. We define $\mathbf{v}^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ and $p^{n+1} : \Omega_{n+1} \to \mathbb{R}^2$ by:

$$\mathbf{v}^{n+1}(\mathbf{x}) = \widehat{\mathbf{v}}^{n+1}(\widehat{\mathbf{x}}),\ \ p^{n+1}(\mathbf{x}) = \widehat{p}^{n+1}(\widehat{\mathbf{x}}),\ \forall \widehat{\mathbf{x}} \in \Omega_n \text{ and } \mathbf{x} = \mathbb{T}_n(\widehat{\mathbf{x}}).$$

Numerical results. Blood flow in large arteries

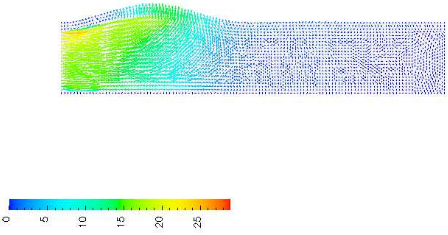
Fluid: length $L = 6\ cm$, height $H = 1\ cm$, viscosity $\mu^F = 0.035\ \frac{g}{cm \cdot s}$, density $\rho^F = 1\ \frac{g}{cm^3}$.
Structure: thickness $h^S = 0.1\ cm$, Young modulus $E = 3 \cdot 10^6\ \frac{g}{cm \cdot s^2}$, Poisson ratio $\nu^S = 0.3$, density $\rho_0^S = 1.1\ \frac{g}{cm^3}$.

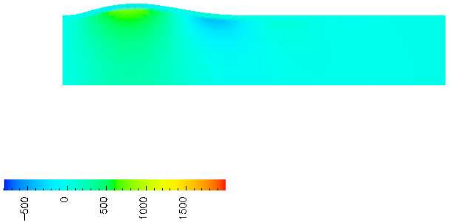
The prescribed boundary stress at the inlet is

$$\mathbf{h}_{in}(\mathbf{x}, t) = \begin{cases} \left(10^3(1 - \cos(2\pi t/0.025)),\ 0\right), & 0 \le t \le 0.025 \\ (0,\ 0), & 0.025 \le t \le T \end{cases}$$

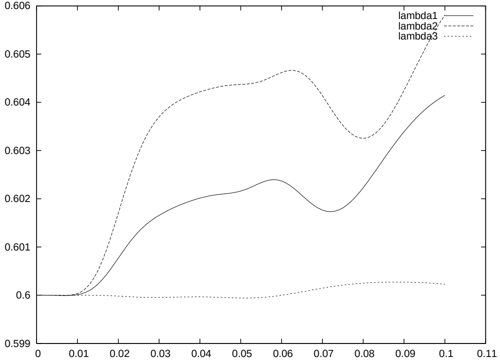
and $\mathbf{h}_{out} = (0,\ 0)$ at the outlet. $\mathbf{g} = (0, 0)^T$.
The numerical tests have been produced using *FreeFem++*.
Numerical parameters: $T = 0.1$, $\Delta t = 0.001,\ 0.0005,\ 0.0025$, $h = 1/20,\ 1/10,\ 1/30$
CPU: **4min13s** using the monolithic approach and **42min** using a partitioned procedures method (BFGS)

Fluid-structure velocities and pressure at time instant $t = 0.025$





Volume of the structure



$$\lambda_1^S = \frac{\nu^S E}{(1-2\nu^S)(1+\nu^S)} = 1730769.23$$
$$\lambda_2^S = 0$$
$$\lambda_3^S = 10^8$$

Conclusions

- Semi-implicit algorithm: the global system of unknowns $\widehat{\mathbf{v}}^{n+1}$, $\widehat{p}^{n+1}$ is implicit, but the domain is computed explicitly.
- The continuity of velocity at the interface is automatically satisfied and the continuity of stress holds in a weak sense.
- The global linear system is solved monolithically.
- The global moving mesh is obtained by gluing the fluid and structure meshes which are matching at the interface. The interface does not pass through the triangles.
- The CPU time is reduced compared to a particular partition procedures strategy.