Topological optimization and minimal compliance in linear elasticity

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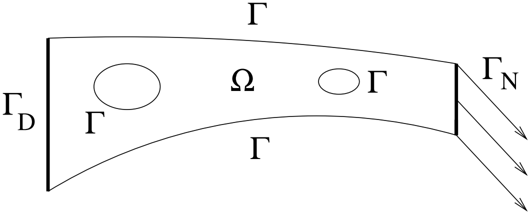
Initial geometrical configuration



Linear elasticity. Notations

$\Omega \subset \mathbb{R}^2$, $\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N \cup \overline{\Gamma}$, $meas(\Gamma_D) > 0$, $meas(\Gamma_N) > 0$,
$meas(\Gamma) > 0$
$\mathbf{y} : \overline{\Omega} \to \mathbb{R}^2$ the displacement
stress tensor in linear elasticity is given by

$$\sigma(\mathbf{y}) = \lambda^S(\nabla \cdot \mathbf{y})\mathbf{I} + 2\mu^S\mathbf{e}(\mathbf{y})$$

where $\lambda^S, \mu^S > 0$ are the Lamé coefficients
$\mathbf{I}$ is the unity matrix and $\mathbf{e}(\mathbf{y}) = \frac{1}{2}\left(\nabla\mathbf{y} + (\nabla\mathbf{y})^T\right)$
given volume load $\mathbf{f} : \Omega \to \mathbb{R}^2$ and surface load $\mathbf{h} : \Gamma_N \to \mathbb{R}^2$
$\mathbf{n}$ is the unit outer normal vector along the boundary

Linear elasticity equations

Find $\mathbf{y} : \overline{\Omega} \to \mathbb{R}^2$ such that

$$\begin{aligned} -\nabla \cdot \sigma(\mathbf{y}) &= \mathbf{f}, \text{ in } \Omega && (1) \\ \mathbf{y} &= 0, \text{ on } \Gamma_D && (2) \\ \sigma(\mathbf{y})\mathbf{n} &= \mathbf{h}, \text{ on } \Gamma_N && (3) \\ \sigma(\mathbf{y})\mathbf{n} &= 0, \text{ on } \Gamma && (4) \end{aligned}$$

The weak formulation is: find $\mathbf{y} \in V$ such that

$$\int_\Omega \sigma(\mathbf{y}) : \nabla \mathbf{v}\, d\mathbf{x} = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v}\, ds, \quad \forall \mathbf{v} \in V \qquad (5)$$

where $\mathbf{f} \in \left(L^2(\Omega)\right)^2$, $\mathbf{h} \in \left(L^2(\Gamma_N)\right)^2$,

$$V = \{\mathbf{v} \in \left(H^1(\Omega)\right)^2 ;\ \mathbf{v} = 0 \text{ on } \Gamma_D\}.$$

Compliance

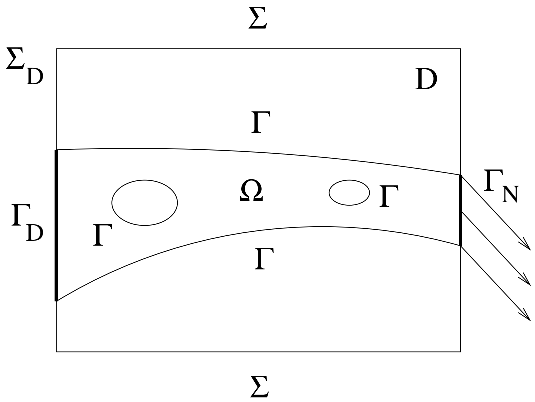
A classical problem in structural design is to find a domain Ω that minimizes the compliance (the work done by the load, expressed by the right-hand side in (5) with $\mathbf{v} = \mathbf{y}$) subject to $\Gamma_N \subset \partial\Omega$, $\Gamma_D \subset \partial\Omega$ and the volume of Ω is prescribed.
In practice, we penalize the volume of Ω, and the function to minimize is

$$\int_\Omega \mathbf{f} \cdot \mathbf{y} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{y} \, ds + \ell \int_\Omega 1 \, d\mathbf{x} \tag{6}$$

where $\ell > 0$ is a penalization coefficient.

Shape optimization problem in fixed domain

$D \subset \mathbb{R}^2$, including the unknown domain Ω, with
$\partial D = \overline{\Sigma}_D \cup \overline{\Gamma}_N \cup \overline{\Sigma}$, such that $\Gamma_D \subset \Sigma_D$



Parametrization

Let $X(D)$ denote a cone of $\mathcal{C}(\overline{D})$. With any $g \in X(D)$, that we call a parametrization, we associate the open set

$$\Omega_g = int\left\{\mathbf{x} \in D;\ g(\mathbf{x}) \geq 0\right\}.$$

We define the family of admissible domains as the connected components of all Ω_g, $g \in X(D)$ satisfying $\Gamma_N \subset \partial\Omega_g$, $\Gamma_D \subset \partial\Omega_g$. We use the following regularization of the Heaviside function

$$H^\epsilon(r) = \begin{cases} 1 - \frac{1}{2}e^{-\frac{r}{\epsilon}}, & r \geq 0, \\ \frac{1}{2}e^{\frac{r}{\epsilon}}, & r < 0, \end{cases} \qquad (7)$$

We have that $H^\epsilon(g)$ is a regularization of the characteristic function of $\overline{\Omega}_g$.

Optimal control problem

For given $\mathbf{f} \in \left(L^2(D)\right)^2$, $\mathbf{h} \in \left(L^2(\Gamma_N)\right)^2$ and $\ell > 0$, we introduce the control problem (with control g) that approximates the shape optimization problem

$$\inf_{g \in X(D)} J(g) = \int_D H^\epsilon(g)\mathbf{f} \cdot \mathbf{y}^\epsilon(g)\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{y}^\epsilon(g)\, ds + \ell \int_D H^\epsilon(g)\, d\mathbf{x} \tag{8}$$

where $\mathbf{y}^\epsilon(g) \in W$ is the solution of

$$\int_D H^\epsilon(g)\sigma\left(\mathbf{y}^\epsilon(g)\right) : \nabla \mathbf{v}\, d\mathbf{x} = \int_D H^\epsilon(g)\mathbf{f} \cdot \mathbf{v}\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v}\, ds, \tag{9}$$

for all $\mathbf{v} \in W$, where

$$W = \{\mathbf{v} \in \left(H^1(D)\right)^2;\ \mathbf{v} = 0 \text{ on } \Sigma_D\}.$$

Existence and uniqueness of the state equation

We set

$$X(D) = \{g \in \mathcal{C}(\overline{D});\ g(\mathbf{x}) = 0,\ \mathbf{x} \in \Gamma_N \cup \Gamma_D\}$$

which is a subspace in $\mathcal{C}(\overline{D})$.

Proposition

The problem (9) has a unique solution $\mathbf{y}^\epsilon(g) \in W$ *and*

$$\|\mathbf{y}^\epsilon(g)\|_{1,D} \leq \frac{C}{c(\epsilon)} \left(\|\mathbf{f}\|_{0,D} + \|\mathbf{h}\|_{0,\Gamma_N} \right) \qquad (10)$$

where $C > 0$ *is independent of* ϵ *and* $c(\epsilon) > 0$ *is indicated below.*

Proof

$$
\begin{aligned}
a(\mathbf{v},\mathbf{v}) &= \int_D H^\epsilon(g)\sigma(\mathbf{v}) : \nabla\mathbf{v}\, d\mathbf{x} \\
&= \int_D H^\epsilon(g)\left(\lambda^S(\nabla\cdot\mathbf{v})^2 + 2\mu^S\mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v})\right) d\mathbf{x} \\
&\geq c(\epsilon)\int_D \left(\lambda^S(\nabla\cdot\mathbf{v})^2 + 2\mu^S\mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v})\right) d\mathbf{x} \\
&\geq c(\epsilon)\int_D 2\mu^S\mathbf{e}(\mathbf{v}) : \mathbf{e}(\mathbf{v})\, d\mathbf{x}.
\end{aligned}
$$

$H^\epsilon(g) \geq c(\epsilon) > 0$ in $\overline{D}$. From the Korn's inequality and Lax-Milgram theorem, we get that the problem has a unique solution

$$
\begin{aligned}
\|\mathbf{y}^\epsilon(g)\|_{1,D} &\leq \frac{C}{c(\epsilon)}\left(\|H^\epsilon(g)\mathbf{f}\|_{0,D} + \|\mathbf{h}\|_{0,\Gamma_N}\right) \\
&\leq \frac{C}{c(\epsilon)}\left(\|\mathbf{f}\|_{0,D} + \|\mathbf{h}\|_{0,\Gamma_N}\right)
\end{aligned}
$$

since $0 < H^\epsilon(g) \leq 1$.

Proposition

When $\epsilon \to 0$*, on a subsequence, we have* $\mathbf{y}^\epsilon|_{\Omega_g} \to \mathbf{y}$ *weakly in* $H^1(\Omega_g)$*. Moreover,* $\mathbf{y} \in V$ *and satisfies (1) - (4) in the distributional sense.*

Proof. Let $\mathbf{f_1} \in L^2(D)$ be the extension by 0 of $\mathbf{f} \in L^2(\Omega_g)$.

$$\int_D H^\epsilon(g)\sigma(\mathbf{y}^\epsilon) : \nabla \mathbf{y}^\epsilon \, d\mathbf{x} = \int_D H^\epsilon(g)\mathbf{f_1} \cdot \mathbf{y}^\epsilon \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{y}^\epsilon \, ds.$$

$$\int_D H^\epsilon(g)[\lambda^S(\nabla \cdot \mathbf{y}^\epsilon)^2 + 2\mu^S \mathbf{e}(\mathbf{y}^\epsilon) : \mathbf{e}(\mathbf{y}^\epsilon)] \, d\mathbf{x} = \int_D H^\epsilon(g)\mathbf{f_1} \cdot \mathbf{y}^\epsilon \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{y}^\epsilon \, ds. \quad (11)$$

$1 \geq H^\epsilon(g) \geq 1/2$ in Ω_g.

One can apply the Korn's inequality and establish that $\mathbf{y}^{\epsilon}|\Omega_g$ is bounded in $H^1(\Omega_g)$. On a subsequence, $\mathbf{y}^{\epsilon}|_{\Omega_g} \to \mathbf{y}$ weakly in $H^1(\Omega_g)$. Moreover, $H^{\epsilon}(g) \to H(g)$ in $L^p(D)$, for any $p \geq 1$.
For any test function $\mathbf{v} \in \mathcal{D}(\Omega_g) \subset W$, we pass to the limit in (9) and obtain

$$\int_{\Omega_g} \sigma(\mathbf{y}) : \nabla \mathbf{v} \, d\mathbf{x} = \int_{\Omega_g} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h} \cdot \mathbf{v} \, ds, \ \forall \mathbf{v} \in \mathcal{D}(\Omega_g)$$

Gâteaux differentiability

Proposition

For any g, w in $X(D)$, the mapping $g \to \mathbf{y}^\epsilon(g) \in W$ is Gâteaux differentiable at g and the directional derivative in the direction w, denoted by $\mathbf{z} \in W$, is the unique solution of the problem

$$\int_D H^\epsilon(g)\sigma(\mathbf{z}) : \nabla \mathbf{v} \, d\mathbf{x} = - \int_D (H^\epsilon)'(g) w \, \sigma(\mathbf{y}^\epsilon(g)) : \nabla \mathbf{v} \, d\mathbf{x} + \int_D (H^\epsilon)'(g) w \, \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x}, \ \forall \mathbf{v} \in W. \tag{12}$$

Proof

Let g, w be fixed in $X(D)$ and $\lambda \neq 0$, small.

$$\int_D H^\epsilon(g+\lambda w)\sigma\left(\mathbf{y}^\epsilon(g+\lambda w)\right) : \nabla \mathbf{v}\, d\mathbf{x}$$
$$= \int_D H^\epsilon(g+\lambda w)\mathbf{f}\cdot\mathbf{v}\, d\mathbf{x} + \int_{\Gamma_N} \mathbf{h}\cdot\mathbf{v}\, ds, \quad \forall \mathbf{v} \in W.$$

Subtracting (9) from the above equation, dividing by λ, setting $\mathbf{z}^\epsilon_\lambda = \frac{\mathbf{y}^\epsilon(g+\lambda w)-\mathbf{y}^\epsilon(g)}{\lambda}$, we get

$$\int_D H^\epsilon(g+\lambda w)\sigma\left(\mathbf{z}^\epsilon_\lambda\right) : \nabla \mathbf{v}\, d\mathbf{x}$$
$$= \; -\int_D \frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda}\sigma\left(\mathbf{y}^\epsilon(g)\right) : \nabla \mathbf{v}\, d\mathbf{x}$$
$$+ \int_D \frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda}\mathbf{f}\cdot\mathbf{v}\, d\mathbf{x}, \quad \forall \mathbf{v} \in W. \quad (13)$$

$\frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda}$ converges to $(H^\epsilon)'(g)\, w$ in $\mathcal{C}(\overline{D})$, for $\lambda \to 0$.
We get that

$$\left\| \frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda} \right\|_{\mathcal{C}(\overline{D})} \leq M, \quad \forall |\lambda| < \lambda_1(\delta),\ \lambda \neq 0 \quad (14)$$

where $M = M(\epsilon)$ is independent of λ, but depends on ϵ.

$$\begin{aligned} \|\mathbf{z}_\lambda^\epsilon\|_{1,D} &\leq \frac{C\,M}{c(\epsilon)} \left(\|\sigma\left(\mathbf{y}^\epsilon(g)\right)\|_{0,D} + \|\mathbf{f}\|_{0,D} \right) \\ &\leq \frac{C\,M}{c(\epsilon)} \left(C_1 \,\|\mathbf{y}^\epsilon(g)\|_{1,D} + \|\mathbf{f}\|_{0,D} \right) \qquad (15) \end{aligned}$$

where $C_1 > 0$ is independent of λ, ϵ such that
$\|\sigma\left(\mathbf{v}\right)\|_{0,D} \leq C_1 \,\|\mathbf{v}\|_{1,D}$, for all $\mathbf{v} \in W$.
Let $\widetilde{\mathbf{z}} \in W$ such that, on a subsequence $\mathbf{z}_\lambda^\epsilon$ converges weakly to $\widetilde{\mathbf{z}}$ in W and strongly in $\left(L^2(D)\right)^2$.

For passing to the limit on a subsequence in (13), we use the Lemma: if $a, a_n \in L^\infty(D)$, $\|a_n\|_{0,\infty,D} \le M$, $a_n \to a$ almost everywhere in D, $b_n \to b$ weakly in $L^2(D)$ and $h \in L^2(D)$, then

$$\lim_{n\to\infty} \int_D a_n b_n h \, dx = \int_D a \, b \, h \, dx.$$

We can apply this Lemma for $a_n = H^\epsilon(g + \lambda_n w)$, $b_n = \sigma\left(\mathbf{z}^\epsilon_{\lambda_n}\right)$ and $h = \nabla \mathbf{v}$.

By passing to the limit on a subsequence in (13) we get that $\widetilde{\mathbf{z}}$ is solution of (12). But, as in Proposition 1, we can show that the problem (12) has a unique solution, then $\widetilde{\mathbf{z}} = \mathbf{z}$ and $\mathbf{z}^\epsilon_\lambda$ converges to $\mathbf{z}$ for $\lambda \to 0$ without taking subsequence, weakly in W and strongly in $\left(L^2(D)\right)^2$.

We prove, also, that $\mathbf{z}^\epsilon_\lambda$ converges to $\mathbf{z}$ strongly in W, but ϵ is fixed.

Directional derivative

Proposition

The directional derivative of the objective function (8) has the form

$$J'(g)w = \int_D H^\epsilon(g)\mathbf{f}\cdot\mathbf{z}\,d\mathbf{x} + \int_D (H^\epsilon)'(g)\,w\,\mathbf{f}\cdot\mathbf{y}^\epsilon(g)\,d\mathbf{x} + \int_{\Gamma_N}\mathbf{h}\cdot\mathbf{z}\,ds + \ell\int_D (H^\epsilon)'(g)\,w\,d\mathbf{x} \quad (16)$$

for any g, w in $X(D)$.

Proof

$$\frac{J(g+\lambda w)-J(g)}{\lambda}$$

$$= \frac{1}{\lambda}\int_D \left(H^\epsilon(g+\lambda w)\mathbf{f}\cdot\mathbf{y}^\epsilon(g+\lambda w) - H^\epsilon(g)\mathbf{f}\cdot\mathbf{y}^\epsilon(g)\right)d\mathbf{x}$$

$$+ \int_{\Gamma_N}\mathbf{h}\cdot\frac{\mathbf{y}^\epsilon(g+\lambda w)-\mathbf{y}^\epsilon(g)}{\lambda}\,ds + \ell\int_D \frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda}\,d\mathbf{x}$$

Subtracting and adding $\int_D H^\epsilon(g+\lambda w)\mathbf{y}^\epsilon(g) : \nabla \mathbf{v}\, d\mathbf{x}$, we get

$$\begin{aligned}
& \frac{1}{\lambda}\int_D (H^\epsilon(g+\lambda w)\mathbf{y}^\epsilon(g+\lambda w) - H^\epsilon(g+\lambda w)\mathbf{y}^\epsilon(g))\cdot \mathbf{f}\, d\mathbf{x} \\
& +\frac{1}{\lambda}\int_D (H^\epsilon(g+\lambda w)\mathbf{y}^\epsilon(g) - H^\epsilon(g)\mathbf{y}^\epsilon(g))\cdot \mathbf{f}\, d\mathbf{x} \\
= \quad & \int_D H^\epsilon(g+\lambda w)\frac{\mathbf{y}^\epsilon(g+\lambda w)-\mathbf{y}^\epsilon(g)}{\lambda}\cdot \mathbf{f}\, d\mathbf{x} \\
& +\int_D \frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda}\mathbf{y}^\epsilon(g)\cdot \mathbf{f}\, d\mathbf{x}
\end{aligned}$$

$H^\epsilon(g+\lambda w)$ converges uniformly to $H^\epsilon(g)$ in $\mathcal{C}(\overline{D})$
$\mathbf{z}^\epsilon_\lambda$ converges strongly to $\mathbf{z}$ in W
$\frac{H^\epsilon(g+\lambda w)-H^\epsilon(g)}{\lambda}$ converges uniformly to $(H^\epsilon)'(g)\, w$ in $\mathcal{C}(\overline{D})$

Directional derivative without using an adjoint system

Proposition

For any g, w in $X(D)$, we have

$$J'(g)w = \int_D (H^\epsilon)'(g)w\,[2\mathbf{f}\cdot\mathbf{y}^\epsilon(g) + \ell - \sigma(\mathbf{y}^\epsilon(g)) : \nabla\mathbf{y}^\epsilon(g)]\,d\mathbf{x}, \tag{17}$$

Proof.

From (9), we put $\mathbf{v} = \mathbf{z} \in W$ and using $\sigma(\mathbf{y}^\epsilon(g)) : \nabla\mathbf{z} = \sigma(\mathbf{z}) : \nabla\mathbf{y}^\epsilon(g)$ we get

$$\int_D H^\epsilon(g)\sigma(\mathbf{z}) : \nabla\mathbf{y}^\epsilon(g)d\mathbf{x} = \int_D H^\epsilon(g)\,\mathbf{f}\cdot\mathbf{z}\,d\mathbf{x} + \int_{\Gamma_N}\mathbf{h}\cdot\mathbf{z}\,ds.$$

Putting $\mathbf{v} = \mathbf{y}^\epsilon(g)$ in (12), it follows

$$\begin{aligned}
&\int_D H^\epsilon(g)\sigma(\mathbf{z}) : \nabla\mathbf{y}^\epsilon(g)\,d\mathbf{x} \\
= \; & -\int_D (H^\epsilon)'(g)w\,\sigma(\mathbf{y}^\epsilon(g)) : \nabla\mathbf{y}^\epsilon(g)\,d\mathbf{x} + \int_D (H^\epsilon)'(g)w\,\mathbf{f}\cdot\mathbf{y}^\epsilon(g)\,d\mathbf{x}.
\end{aligned}$$

Some descent directions

$$
\begin{aligned}
d &= 2\mathbf{f}\cdot\mathbf{y}^{\epsilon}(g)+\ell-\sigma\left(\mathbf{y}^{\epsilon}(g)\right):\nabla\mathbf{y}^{\epsilon}(g)\\
R(r) &= \begin{cases} c(-1+e^{r}), & r<0,\\ c(1-e^{-r}), & r\geq 0\end{cases}
\end{aligned}
\tag{18}
$$

where $c>0$. $r\,R(r)\geq 0$ for all $r\in\mathbb{R}$.

Proposition

The following are descent directions for the objective function $J(g)$:

$$
i)\quad w_d = -H^{\epsilon}(g)d \tag{19}
$$

$$
ii)\quad w_d = -H^{\epsilon}(g)R(d) \tag{20}
$$

$$
iii)\quad w_d = -\widetilde{d} \tag{21}
$$

under the assumption that $w_d\in X(D)$. At iii), $\widetilde{d}\in H^1(D)$ is the solution of

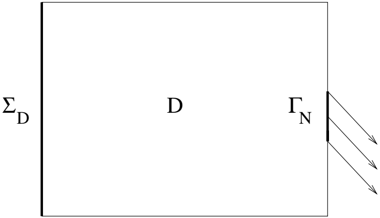
$$
\int_D \gamma(\nabla\widetilde{d}\cdot\nabla v)+\widetilde{d}\,v\,d\mathbf{x} = \int_D (H^{\epsilon})'(g)d\,v\,d\mathbf{x},\quad \forall v\in H^1(D) \tag{22}
$$

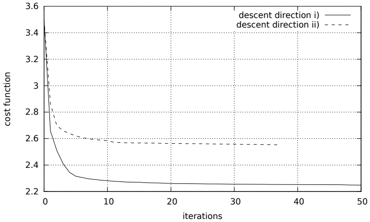
Numerical examples

We have employed the software FreeFem++.
The dimensions and the starting domains are from the web site of the team directed by G. Allaire, the files `levelset-cantilever.edp` and `pont.homog.struct.edp`.

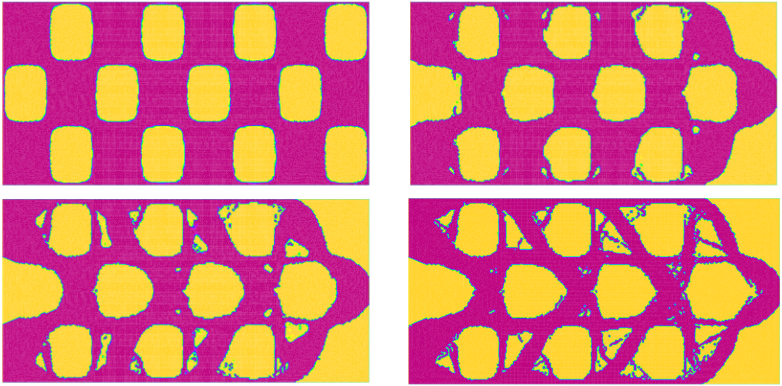
Example 1. Cantilever

$D =]0,2[\times]-0.5,0.5[$, $\Sigma_D = \{0\}\times]-0.5,0.5[$,
$\Gamma_N = \{2\}\times]-0.1,0.1[$, Lamé coefficients $\lambda^S = 1$, $\mu^S = 8$,
$\mathbf{f} = (0,0)$, $\mathbf{h} = (0,-5)$, $\ell = 0.5$

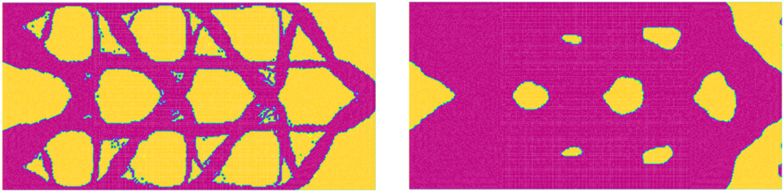




Left: Geometrical configuration of D. Right: Convergence history of the objective functions for descent directions i) and ii)



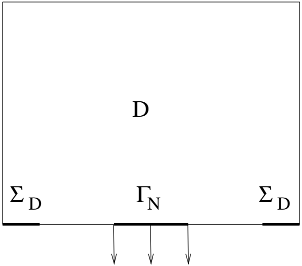
domains using descent direction *i*) (50 iterations)

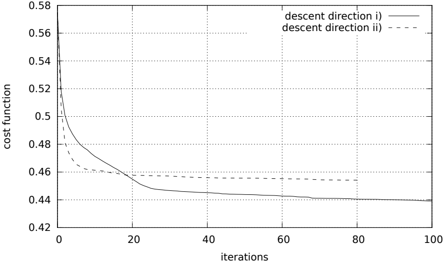


Optimal domains using descent directions *ii*) (left, after 38 iterations) and *iii*) (right, after 3 iterations)

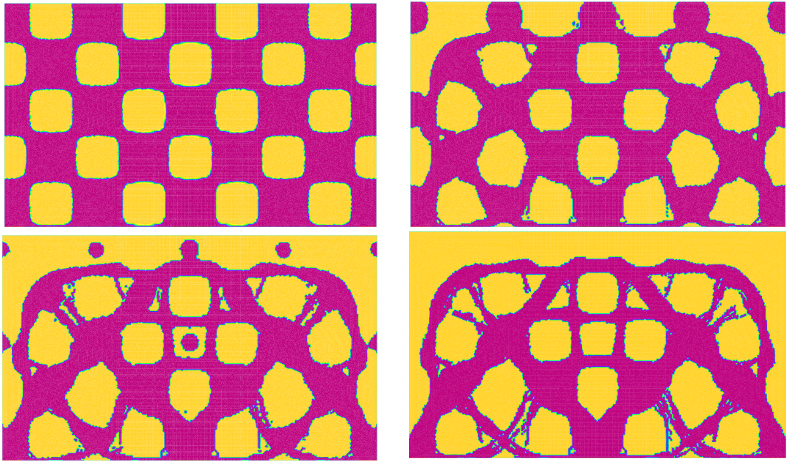
Example 2. Bridge

$D =]-1, 1[\times]0, 1.2[$, $\Sigma_D = (]-1, -0.9[\cup]0.9, 1[) \times \{0\}$, $\Gamma_N =]-0.1, 0.1[\times \{0\}$, Young modulus $E = 1$, Poisson ratio $\nu = 0.3$, $\mathbf{f} = (0, 0)$, $\mathbf{h} = (0, -1)$, $\ell = 0.1$

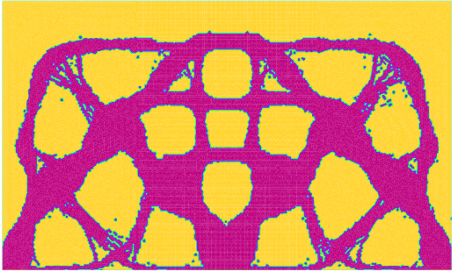




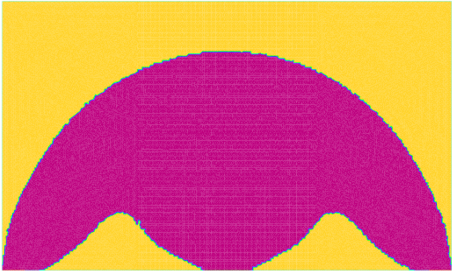
Left: Geometrical configuration of D. Right: Convergence history of the objective functions for descent directions i) and ii).



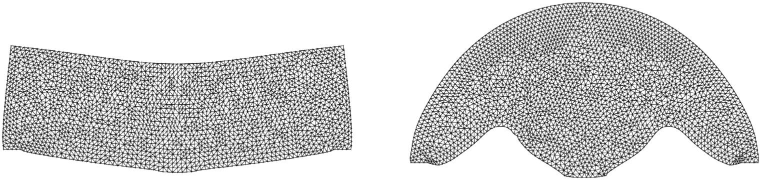
domains using descent direction i) (after 100 iterations)



Optimal domain using descent directions *ii*) after 80 iterations.



Optimal domain using descent directions i) after 100 iterations, for the initial domain $\Omega_0 =]-1,1[\times]0,0.6[$, the bottom half of D.



The cost decreases from 0.378632 (left image) to 0.297857 (right image).

Merci !