Approximation of a fluid-structure interaction problem using fictitious domain approach with penalization

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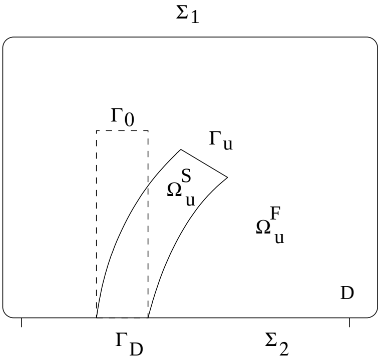
Geometrical configuration



$$\mathbf{x} = \varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$$

The deformed domain $\Omega_u^S = \varphi\left(\Omega_0^S\right)$.

Strong formulation

Find $\mathbf{u} : \overline{\Omega}_0^S \to \mathbb{R}^2$, $\mathbf{v} : \overline{\Omega}_u^F \to \mathbb{R}^2$ and $p : \overline{\Omega}_u^F \to \mathbb{R}$ such that

$$
\begin{aligned}
-\nabla_{\mathbf{X}} \cdot \sigma^S(\mathbf{u}) &= \mathbf{f}^S, \quad \text{in } \Omega_0^S && (1)\\
\mathbf{u} &= 0, \quad \text{on } \Gamma_D && (2)\\
-\nabla \cdot \sigma^F(\mathbf{v}, p) &= \mathbf{f}^F, \quad \text{in } \Omega_u^F && (3)\\
\nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_u^F && (4)\\
\mathbf{v} &= \mathbf{g}, \quad \text{on } \Sigma_1 && (5)\\
\mathbf{v} &= 0, \quad \text{on } \Sigma_2 \setminus \Gamma_D && (6)\\
\mathbf{v} &= 0, \quad \text{on } \Gamma_u && (7)\\
\omega\left(\sigma^F(\mathbf{v}, p)\,\mathbf{n}^F\right) \circ \varphi &= -\sigma^S(\mathbf{u})\,\mathbf{n}^S, \quad \text{on } \Gamma_0 && (8)
\end{aligned}
$$

Fictitious domain approach using penalization

$$\begin{aligned} -\nabla \cdot \sigma^F(\mathbf{v}, p) + \frac{1}{\epsilon}\mathcal{P}(\mathbf{v}) &= \mathbf{f}^F, \quad \text{in } \Omega_u^S \\ \nabla \cdot \mathbf{v} &= 0, \quad \text{in } \Omega_u^S \end{aligned}$$

where $\epsilon > 0$ is a penalization parameter,

$$\mathcal{P}(\mathbf{v}) = \left(|v_1|^{\alpha-1} sgn(v_1), |v_2|^{\alpha-1} sgn(v_2)\right)$$

where $\mathbf{v} = (v_1, v_2)$ and $1 < \alpha < 2$ is a real number.

$$\chi_u^S(\mathbf{x}) = \begin{cases} 1, & \mathbf{x} \in \overline{\Omega}_u^S \\ 0, & \mathbf{x} \in D \setminus \overline{\Omega}_u^S \end{cases} \quad \text{and} \quad \chi_u^F = 1 - \chi_u^S.$$

$$\begin{aligned} -\nabla \cdot \sigma^F(\mathbf{v}, p) + \frac{1}{\epsilon}\chi_u^S \mathcal{P}(\mathbf{v}) &= \mathbf{f}^F, \quad \text{in } D \\ \nabla \cdot \mathbf{v} &= 0, \quad \text{in } D. \end{aligned}$$

Weak formulation in the structure domain (I)

Structure equations in the initial domain

$$\int_{\Omega_0^S} \sigma^S(\mathbf{u}) : \nabla_{\mathbf{X}} \mathbf{w}^S \, d\mathbf{X} = \int_{\Omega_0^S} \mathbf{f}^S \cdot \mathbf{w}^S \, d\mathbf{X} + \int_{\Gamma_0} \sigma^S(\mathbf{u}) \, \mathbf{n}^S \cdot \mathbf{w}^S dS.$$

Fluid equations in the deformed structure domain

We define $\widetilde{\mathbf{w}}^S : \Omega_u^S \to \mathbb{R}^2$, $\widetilde{\mathbf{w}}^S = \mathbf{w}^S \circ \varphi^{-1}$

$$\int_{\Omega_u^S} \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) : \nabla \widetilde{\mathbf{w}}^S \, d\mathbf{x} + \frac{1}{\epsilon} \int_{\Omega_u^S} \mathcal{P}(\mathbf{v}_\epsilon) \cdot \widetilde{\mathbf{w}}^S \, d\mathbf{x}$$
$$= \int_{\Omega_u^S} \mathbf{f}^F \cdot \widetilde{\mathbf{w}}^S \, d\mathbf{x} - \int_{\Gamma_u} \sigma^F(\mathbf{v}_\epsilon, p_\epsilon) \, \mathbf{n}^F \cdot \widetilde{\mathbf{w}}^S \, ds$$

Weak formulation in the structure domain (II)

Fluid equations in the undeformed structure domain

$$\int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S\, d\mathbf{X} + \frac{1}{\epsilon}\int_{\Omega_0^S} J\mathcal{P}(\mathbf{v}_\epsilon\circ\varphi)\cdot\mathbf{w}^S\, d\mathbf{X}$$
$$= \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\, d\mathbf{X} - \int_{\Gamma_0}\omega\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\,\mathbf{n}^F\circ\varphi\right)\cdot\mathbf{w}^S\, dS.$$

Subtracting from the structure equations, we get

$$\int_{\Omega_0^S}\sigma^S(\mathbf{u}) : \nabla_{\mathbf{X}}\mathbf{w}^S\, d\mathbf{X} - \int_{\Omega_0^S}\mathbf{f}^S\cdot\mathbf{w}^S\, d\mathbf{X}$$
$$= \int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S\, d\mathbf{X}$$
$$+\frac{1}{\epsilon}\int_{\Omega_0^S} J\mathcal{P}(\mathbf{v}_\epsilon\circ\varphi)\cdot\mathbf{w}^S\, d\mathbf{X} - \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\, d\mathbf{X}$$

Constitutive relations

$$\epsilon(\mathbf{w}) = \frac{1}{2}\left(\nabla \mathbf{w} + (\nabla \mathbf{w})^T\right).$$

Linear elasticity

$$a_S\left(\mathbf{u}, \mathbf{w}^S\right) = \int_{\Omega_0^S} \left(\lambda^S (\nabla \cdot \mathbf{u}) \left(\nabla \cdot \mathbf{w}^S\right) + 2\mu^S \epsilon(\mathbf{u}) : \epsilon\left(\mathbf{w}^S\right)\right) \, d\mathbf{X}.$$

Stokes equations

$$\begin{aligned} a_F(\mathbf{v}, \mathbf{w}) &= \int_D 2\mu^F \epsilon(\mathbf{v}) : \epsilon(\mathbf{w}) \, d\mathbf{x} \\ b_F(\mathbf{w}, p) &= -\int_D (\nabla \cdot \mathbf{w}) \, p \, d\mathbf{x}. \end{aligned}$$

Parametrization of the characteristic function

Let $j \in W^{1,\infty}(D)$ be a parametrization of $\Omega_0^S \subset D$, i.e. :

$$\begin{array}{rcl} j(\mathbf{x}) & > & 0,\ \mathbf{x} \in \Omega_0^S, \\ j(\mathbf{x}) & < & 0,\ \mathbf{x} \in D \setminus \overline{\Omega}_0^S, \\ j(\mathbf{x}) & = & 0,\ \mathbf{x} \in \partial\Omega_0^S. \end{array}$$

$\Omega_u^S = \varphi(\Omega_0^S)$, where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$

$$j_u(\mathbf{y}) = \begin{cases} j(\mathbf{x}), & \mathbf{y} = \varphi(\mathbf{x}) \in \Omega_u^S \\ 0, & \mathbf{y} \in \partial\Omega_u^S \\ -dist(\mathbf{y}, \overline{\Omega}_u^S), & \mathbf{y} \notin \overline{\Omega}_u^S \end{cases}$$

is a parametrization of Ω_u^S, $j_u \in W^{1,\infty}(D)$.

Regularization of the characteristic function

If H is the Heaviside function $H : \mathbb{R} \to \{0, 1\}$,

$$H(r) = \begin{cases} 1, & r \geq 0 \\ 0, & r < 0 \end{cases}$$

then $H(j_u(\cdot))$ is the characteristic function of Ω_u^S.

$$\Omega_0^\epsilon \subset\subset \Omega_0^S, \quad \Omega_u^\epsilon = (\mathbf{id} + \mathbf{u})\,(\Omega_0^\epsilon)$$

There exits $\mu_\epsilon > 0$ such that $j(\mathbf{x}) \geq \mu_\epsilon > 0$, for all $\mathbf{x} \in \Omega_0^\epsilon$.
Then we take $\widetilde{H} = H^{\mu_\epsilon}$, the Yosida regularization of H

$$H^{\mu_\epsilon}(r) = \begin{cases} 1, & r \geq \mu_\epsilon \\ \frac{r}{\mu_\epsilon} & 0 \leq r < \mu_\epsilon \\ 0, & r < 0 \end{cases}$$

It follows that $H^{\mu_\epsilon}(j_u(\mathbf{x})) = 1$ for all $\mathbf{x} \in \Omega_u^\epsilon$.
$\widetilde{H}(j_u)$ is Lipschitz and $0 \leq \widetilde{H}(j_u(\mathbf{x})) \leq 1$ for all $\mathbf{x} \in \overline{D}$.

Functional spaces

$$
\begin{aligned}
W^S &= \left\{ \mathbf{w}^S \in \left(H^1\left(\Omega_0^S\right)\right)^2;\ \mathbf{w}^S = 0 \text{ on } \Gamma_D \right\}, \\
W &= \left(H_0^1(D)\right)^2, \\
Q &= L_0^2(D) = \{q \in L^2(D);\ \int_D q\, dx = 0\}.
\end{aligned}
$$

$\mathbf{f}^F \in \left(L^2(D)\right)^2$, $\mathbf{f}^S \in \left(L^2(\Omega_0^S)\right)^2$, $\mathbf{g} \in \left(H^{1/2}(\partial D)\right)^2$, such that $\mathbf{g} = 0$ on Σ_2 and $\int_{\Sigma_1} \mathbf{g} \cdot \mathbf{n}^F\, ds = 0$.

For a given $\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2$, such that $\|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1$ and $\mathbf{u} = 0$ on Γ_D, we define:

- fluid velocity $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on ∂D,
- fluid pressure $p_\epsilon \in Q$,
- structure displacement $\mathbf{u}_\epsilon \in W^S$

Coupled system

$$
\begin{aligned}
a_F(\mathbf{v}_\epsilon, \mathbf{w}) + b_F(\mathbf{w}, p_\epsilon) & \\
+\frac{1}{\epsilon}\int_D \widetilde{H}(j_u)\mathcal{P}(\mathbf{v}_\epsilon)\cdot\mathbf{w}\,dx &= \int_D \mathbf{f}^F\cdot\mathbf{w}\,d\mathbf{x}, \quad \forall \mathbf{w}\in W \qquad (9)\\
b_F(\mathbf{v}_\epsilon, q) &= 0, \quad \forall q\in Q \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
a_S\left(\mathbf{u}_\epsilon, \mathbf{w}^S\right) &= \int_{\Omega_0^S} \mathbf{f}^S\cdot\mathbf{w}^S\,d\mathbf{X}\\
&+ \int_{\Omega_0^S} J\left(\sigma^F(\mathbf{v}_\epsilon, p_\epsilon)\circ\varphi\right)\mathbf{F}^{-T} : \nabla_{\mathbf{X}}\mathbf{w}^S\,d\mathbf{X}\\
&+ \frac{1}{\epsilon}\int_{\Omega_0^S} J\widetilde{H}(j_u\circ\varphi)\mathcal{P}(\mathbf{v}_\epsilon\circ\varphi)\cdot\mathbf{w}^S\,d\mathbf{X}\\
&- \int_{\Omega_0^S} J\left(\mathbf{f}^F\circ\varphi\right)\cdot\mathbf{w}^S\,d\mathbf{X}, \ \forall \mathbf{w}^S\in W^S \qquad (11)
\end{aligned}
$$

where $\varphi(\mathbf{X}) = \mathbf{X} + \mathbf{u}(\mathbf{X})$, $\mathbf{F}(\mathbf{X}) = \mathbf{I} + \nabla_{\mathbf{X}}\mathbf{u}(\mathbf{X})$, $J(\mathbf{X}) = \det \mathbf{F}(\mathbf{X})$.

Definition of the nonlinear operator

$$\left\{\mathbf{u} \in \left(W^{1,\infty}(\Omega_0^S)\right)^2;\ \|\mathbf{u}\|_{1,\infty,\Omega_0^S} < 1,\ \mathbf{u} = 0 \text{ on } \Gamma_D\right\}$$

Fluid problem

$$\mathbf{u} \to \mathbf{v}_\epsilon \in \widetilde{\mathbf{g}} + \left(H_0^1\left(D\right)\right)^2 \text{ and } p_\epsilon \in L_0^2\left(D\right)$$

Structure problem

$$\mathbf{u},\ \mathbf{v}_\epsilon,\ p_\epsilon \to \mathbf{u}_\epsilon \in \left\{\mathbf{w}^S \in \left(H^1\left(\Omega_0^S\right)\right)^2;\ \mathbf{w}^S = 0 \text{ on } \Gamma_D\right\}$$

Nonlinear operator

$$T_\epsilon(\mathbf{u}) = \mathbf{u}_\epsilon.$$

Existence of the penalized fluid-structure problem

$$B_\delta = \{u \in W^{1,\infty}(\Omega_0^S)^2; ||u||_{1,\infty,\Omega_0^S} \leq \eta_\delta,\ u = 0 \text{ on } \Gamma_D\}$$

Theorem

Let $\epsilon > 0$ fixed. If f^F, f^S and g are "small" in their own norms, then the nonlinear operator T_ϵ has at least one fixed point $\mathbf{u}_\epsilon$ in B_δ.

See: A. Halanay, C.M. Murea, D. Tiba, Existence and approximation for a steady fluid-structure interaction problem using fictitious domain approach with penalization, in progress

Problem: What happens when $\epsilon \to 0$?

Partitioned procedures based on the fixed point iterations

The penalized term $\mathcal{P}(\mathbf{v}) = \left(|v_1|^{\alpha-1} sgn(v_1), |v_2|^{\alpha-1} sgn(v_2)\right)$, where $1 < \alpha < 2$ is non-linear in $\mathbf{v}$. But, if α is close to 2, we can approach $\mathcal{P}(\mathbf{v})$ by $\mathbf{v}$. We can also replace $\widetilde{H}(j_u)$ by the characteristic function χ_u^S.

Algorithm

Step 1. Given the initial displacement of the structure $\mathbf{u}^0 \in W^S$, compute the characteristic function $\chi_{u^0}^S$, put $k := 0$

Step 2. Find the velocity $\mathbf{v}_\epsilon \in \left(H^1(D)\right)^2$, $\mathbf{v}_\epsilon = \mathbf{g}$ on Σ_1, $\mathbf{v}_\epsilon = 0$ on Σ_2 and the pressure $p_\epsilon^k \in Q$ by solving the fluid problem

$$a_F\left(\mathbf{v}_\epsilon^k, \mathbf{w}\right) + b_F\left(\mathbf{w}, p_\epsilon^k\right) + \frac{1}{\epsilon}\int_D \chi_{u_\epsilon^k}^S \mathbf{v}_\epsilon^k \cdot \mathbf{w}\, d\mathbf{x} = \int_D \mathbf{f}^F \cdot \mathbf{w}\, d\mathbf{x},\ \forall \mathbf{w} \in W$$

$$b_F\left(\mathbf{v}_\epsilon^k, q\right) = 0,\ \forall q \in Q$$

Step 3. Find the new displacement of the structure $\mathbf{u}_\epsilon^{k+1} \in W^S$ by solving

$$a_S\left(\mathbf{u}_\epsilon^{k+1}, \mathbf{w}^S\right) = \int_{\Omega_0^S} \left(\mathbf{f}^S - \mathbf{f}^F\right) \cdot \mathbf{w}^S \, d\mathbf{x} + \int_{\Omega_0^S} 2\mu^F \epsilon\left(\mathbf{v}_\epsilon^k\right) : \epsilon\left(\mathbf{w}^S\right) \, d\mathbf{x}$$
$$- \int_{\Omega_0^S} \left(\nabla \cdot \mathbf{w}^S\right) p_\epsilon^k \, d\mathbf{x} + \frac{1}{\epsilon} \int_{\Omega_0^S} \left(\mathbf{v}_\epsilon^k \circ \varphi_\epsilon^k\right) \cdot \mathbf{w}^S \, d\mathbf{x} \quad \forall \mathbf{w}^S \in W^S$$

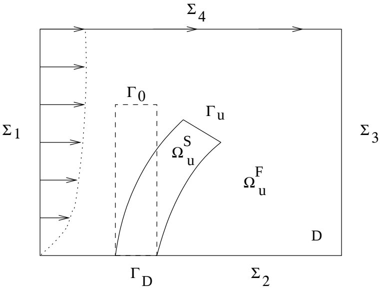
where $\varphi_\epsilon^k(\mathbf{X}) = \mathbf{X} + \mathbf{u}_\epsilon^k(\mathbf{X})$.

Step 4. Stopping test: if $\left\|\mathbf{u}_\epsilon^k - \mathbf{u}_\epsilon^{k+1}\right\|_{0,\Omega_0^S} \leq tol$, then **Stop**

Step 5. Compute the characteristic function $\chi^S_{u_\epsilon^{k+1}}$, put $k := k+1$ and **Go to Step 2.**

Under the assumption of small displacements for the structure, we can approach the Jacobian determinant J by 1 and the gradient of the deformation $\mathbf{F}$ by the identity matrix $\mathbf{I}$.

Numerical test 1. Shell in steady-state cross flow



Physical parameters

Structure: length $\ell = 3\ m$, thickness $h = 0.125\ m$, Young modulus $E^S = 1.6 \times 10^6\ N/m^2$, Poisson's ratio $\nu^S = 0.49$

Fluid: length $L = 12\ m$, width $H = 5\ m$, mass density $\rho^F = 1000\ Kg/m^3$, dynamic viscosity $\mu^F = 0.1\ N \cdot s/m^2$. The velocity boundary conditions at the inflow:

$$v_1(x_1, x_2) = V \times 1.5 \frac{\left(2H\, x_2 - x_2^2\right)}{H^2}\ m/s,\ V = 5, \quad v_2(x_1, x_2) = 0;$$

at the bottom: no-slip; at the top: slip; at the outflow: the tangential velocity and the normal traction are zero

$Re = \frac{V \rho^F H}{\mu^F} = 250000$

Numerical parameters

Fluid mesh: 82512 triangles and 41682 vertices

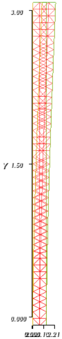
Structure mesh: 188 triangles and 145 vertices

Finite elements: fluid velocity $\mathbb{P}_1$+*bubble*, fluid pressure $\mathbb{P}_1$, structure displacement $\mathbb{P}_1$, characteristic function $\mathbb{P}_0$

$\epsilon = 10^{-3}$, $tol = 10^{-8}$

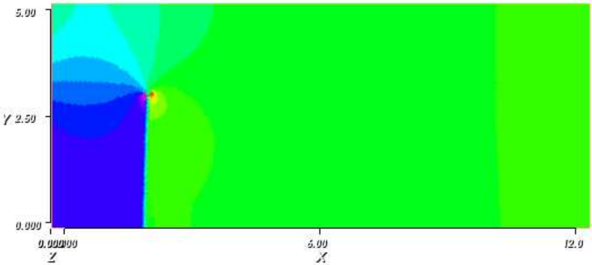
No of iterations of the fixed point algorithm: 7

The initial and the deformed structure mesh



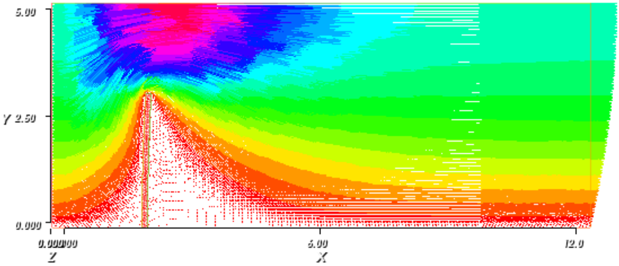
The maximal displacement: 0.083 m

The fluid pressure

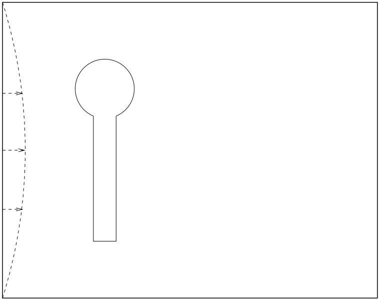


The fluid velocity around the final position of the structure

$$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}} = \sqrt{\int_D \chi^S_{u_\epsilon} \mathbf{v}_\epsilon \cdot \mathbf{v}_\epsilon \, d\mathbf{x}} = 0.047270$$



Numerical test 2. Flexible appendix in a flow



A rectangular flexible appendix is attached to a fixed circle. The circle center is positioned at $(0.2, 0.2)$ m measured from the left top corner of the channel.

Physical parameters

Structure: circle radius $r = 0.5\ m$, length $\ell = 0.35\ m$, thickness $h = 0.02\ m$, Young modulus $E^S = 1.6 \times 10^6\ N/m^2$, Poisson's ratio $\nu^S = 0.49$

Fluid: length $L = 2.5\ m$, width $H = 0.75\ m$, mass density $\rho^F = 1260\ Kg/m^3$, dynamic viscosity $\mu^F = 1.420\ N \cdot s/m^2$. The velocity boundary conditions at the inflow:

$$v_1(x_1, x_2) = V \times 1.5 \frac{(H\,x_2 - x_2^2)}{(H/2)^2}\ m/s,\ \ V = 1, \quad v_2(x_1, x_2) = 0;$$

at the bottom and the top: no-slip; at the outflow: traction free

$Re = \frac{V \rho^F H}{\mu^F} = 665$

Numerical parameters

Fluid mesh: 30330 triangles and 15461 vertices

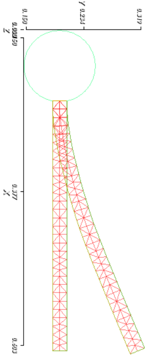
Structure mesh: 128 triangles and 97 vertices

Finite elements: fluid velocity $\mathbb{P}_1$+*bubble*, fluid pressure $\mathbb{P}_1$, structure displacement $\mathbb{P}_1$, characteristic function $\mathbb{P}_0$

$\epsilon = 10^{-4}$, $tol = 10^{-8}$

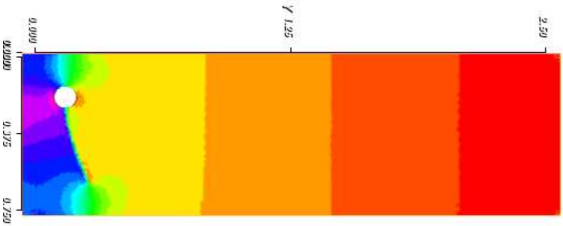
No of iterations of the fixed point algorithm: 8

The initial and the deformed structure mesh



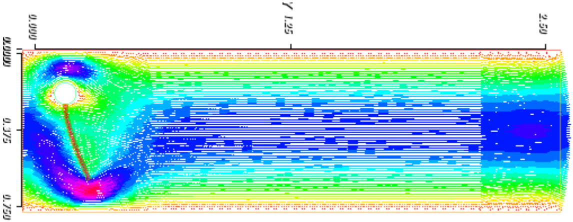
The maximal horizontal displacement: 0.10886 m

The fluid pressure



The fluid velocity around the final position of the structure

$$\|\mathbf{v}_\epsilon\|_{0,\Omega^S_{u_\epsilon}} = \sqrt{\int_D \chi^S_{u_\epsilon} \mathbf{v}_\epsilon \cdot \mathbf{v}_\epsilon \, d\mathbf{x}} = 0.080920$$



Thank you!