A Penalization Method for the Elliptic and Parabolic Obstacle Problem

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Elliptic obstacle problem, strong formulation

Let D be a smooth domain in \mathbb{R}^2 and let $\psi:D\to\mathbb{R}$ denote the obstacle function.

We assume $\psi \in H^2(D)$ and $\psi_{|\partial D} \leq 0$ to be consistent with the homogeneous Dirichlet conditions.

Let $f \in L^2(D)$ be given.

$$-\Delta y = f \text{ in } D^+ = \{ x \in D; \ y(x) > \psi(x) \},$$
 (1)

$$-\Delta y \geq f \text{ in } D \setminus D^+ = \{x \in D; y(x) = \psi(x)\}, \qquad (2)$$

$$y = \psi$$
 and $\frac{\partial y}{\partial n} = \frac{\partial \psi}{\partial n}$ on $\partial D^+ \cap D$, (3)

$$y = 0 \text{ on } \partial D. \tag{4}$$

The set D^+ is called the noncoincidence set, while its complementary is the coincidence set.

Weak formulation

Find $y \in K$ such that

$$\int_{D} \nabla y \cdot (\nabla y - \nabla v) \, dx \leq \int_{D} f(y - v) \, dx, \forall v \in K \quad (5)$$

$$K = \left\{ v \in H_0^1(D); \ v \geq \psi \text{ a.e. in } D \right\}. \quad (6)$$

The problem (5)-(6) is equivalent with the variational problem

$$\min_{v\in K}\left\{\frac{1}{2}\int_{D}\left|\nabla v\right|^{2}dx-\int_{D}f\,v\,dx\right\}.$$
(7)

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Regularity

In the case when $f \in L^2(D)$, $\psi \in H^2(D)$ with the compatibility condition $\psi_{|\partial D} \leq 0$, it is known that the weak solution satisfies the regularity property $y \in H^2(D)$ and the strong formulation may be used. Moreover, in this case, the obstacle problem may be written as a multivalued equation

$$-\Delta y + \beta(y - \psi) \ni f$$
 in D

where $\beta \subset \mathbb{R} \times \mathbb{R}$ is the maximal monotone graph given by

$$\beta(z) = \begin{cases}] -\infty, 0], & z = 0, \\ 0, & z > 0, \\ \emptyset, & z < 0. \end{cases}$$
(8)

Algorithm 1

- 1) Choose n = 0, $\epsilon_0 > 0$, $\Omega_0 \subset D$ open, $\mathbf{y}_{-1} \in H^1_0(D)$;
- 2) Compute $y_n \in H_0^1(D)$ as solution of the linear elliptic equation

$$-\Delta y_n + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_n} (y_n - \psi) = f \text{ in } D$$
(9)

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(here $\chi_{D\setminus\Omega_n}$ is the characteristic function of $D\setminus\Omega_n$, corresponding to the approximation of the coincidence set in iteration *n*); 3) $\mathbf{y}_n = \max\{y_n, \psi\}, \ \Omega_{n+1} = \{x \in D; \ \mathbf{y}_n(x) > \psi(x)\}, \ \epsilon_{n+1} = \frac{\epsilon_n}{2};$ 4) If $\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_{L^2(D)} < tol$ then STOP else n=n+1 GO TO step 2.

We can start with $\Omega_0 = D$ and $\mathbf{y}_{-1} > \psi$.

Remarks

Let $\beta_{\epsilon} : \mathbb{R} \to \mathbb{R}$ denote the Yosida approximation of β . We have $\beta_{\epsilon}(r) = 0$ if r > 0 and $\beta_{\epsilon}(r) = \frac{1}{\epsilon}r$ if $r \leq 0$. Notice that $\beta_{\epsilon}(r) = \beta'_{\epsilon}(r)r$, if $r \neq 0$.

We can rewrite the step 2 of the Algorithm 1 as

$$-\Delta y_n + \left(\beta_{\epsilon_n}'(y_{n-1} - \psi)\right)(y_n - \psi) = f.$$
(10)

The usual approximation by regularization of the variational inequality is

$$-\Delta \tilde{y}_n + \beta_{\epsilon_n} \left(\tilde{y}_n - \psi \right) = f \text{ in } D, \qquad (11)$$

plus homogeneous boundary conditions on ∂D .

The algorithm uses just linear elliptic equations in the whole domain D.

The type of penalization term from Step 2 may be compared with the approach developed in shape optimization problems in Tiba 2009.

A classical penalization term for solving the obstacle problem is $\frac{1}{\epsilon}(y_n-\psi)^-$ where $v^-(x)=-v(x)$ if v(x)<0 and $v^-(x)=0$ if

Two obstacles problem

We denote now $\psi_1, \psi_2 : D \to \mathbb{R}$, $\psi_1 \le \psi_2$ in D, $\psi_1|_{\partial D} \le 0$, $\psi_2|_{\partial D} \ge 0$ and

$$\widetilde{K}=\left\{y\in H^1_0(D); \ \psi_1(x)\leq y(x)\leq \psi_2(x) ext{ a.e. in } D
ight\}$$

which is a closed convex subset of $H_0^1(D)$. To \widetilde{K} , the following variational inequality may be associated:

$$\int_{D} \nabla y \cdot (\nabla y - \nabla v) \, dx \leq \int_{D} f(y - v) \, dx, \quad \forall v \in \widetilde{K}$$
(12)

The existence of a unique solution $y \in \widetilde{K}$ is wellknown.

Algorithm 2

1) Choose n = 0, $\epsilon_0 > 0$, $\Omega_1^0 \subset D$, $\Omega_2^0 \subset D$ open subsets such that $(D \setminus \Omega_1^0) \cap (D \setminus \Omega_2^0) = \emptyset$, \mathbf{y}_{-1} ;

2) Compute $y_n \in H^1_0(D)$ as solution of the linear elliptic equation

$$-\Delta y_n + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_1^n} (y_n - \psi_1) + \frac{1}{\epsilon_n} \chi_{D \setminus \Omega_2^n} (y_n - \psi_2) = f \text{ in } D \quad (13)$$

3) Compute
$$\mathbf{y}_n = \min \{\psi_2, \max\{y_n, \psi_1\}\},\$$

 $\Omega_1^{n+1} = \{x \in D; \mathbf{y}_n(x) > \psi_1(x)\},\$
 $\Omega_2^{n+1} = \{x \in D; \mathbf{y}_n(x) < \psi_2(x)\} \epsilon_{n+1} = \frac{\epsilon_n}{2};\$
4) If $\|\mathbf{y}_n - \mathbf{y}_{n-1}\|_{L^2(D)} < tol$ then STOP else n=n+1 GO TO step 2.

Convergence

Lemma Denote by $\hat{y} \in H^2(D) \cap H^1_0(D)$ the solution of

$$-\Delta \hat{y} = f \text{ in } D, \ \hat{y} = 0 \text{ on } \partial D.$$
(14)

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Then $y \ge \hat{y}$ a. e. in D.

Lemma The solution y of (1)-(4) satisfies the same problem with ψ replaced by $\widehat{\psi}$.

Convergence

For Algorithm 1.

Theorem i) On a subsequence, $y_n \to \check{y}$ weakly in $H_0^1(D)$ and $\Omega_n \to \Omega$ in the complementary Hausdorff-Pompeiu topology. ii) If $\check{y} \in C^1(D)$ and $\{\Omega_n\}$ are uniformly of class C, then \check{y} is the solution of (1)-(4) with $D^+ = \Omega$. The convergence is valid without taking subsequences.

For Algorithm 2.

Theorem The sequence $\{y_n\}$ is bounded in $L^2(D)$. Moreover, there is C > 0, independent of *n*, such that:

$$\int_{D\setminus\Omega_2^n} (y_n-\psi_2)_+^2 dx + \int_{D\setminus\Omega_1^n} (y_n-\psi_1)_-^2 dx \le C\epsilon_n.$$
(15)

Numerical tests

$$D = \{(x_1, x_2); \ \sqrt{x_1^2 + x_2^2} < 1\}$$

$$\psi(x_1, x_2) = \begin{cases} \frac{1}{2}, \ 0.5 \le x_1 \le 0.7 \text{ and } -0.1 \le x_2 \le 0.1 \\ -12.5 \left((x_1 + 0.4)^2 + x_2^2\right) + 0.5, \ (x_1 + 0.4)^2 + x_2^2 \le 0.08 \\ -\frac{1}{2}, \text{ otherwise.} \end{cases}$$

We notice that ψ is not continuous, but the method still works.

$$f(x_1, x_2) = \begin{cases} 3.5, & x_1^2 + x_2^2 \le 0.16 \\ -0.001, & \text{otherwise.} \end{cases}$$

The mesh has 109898 triangles and 55350 vertices, the tolerance for the stopping test $tol = 10^{-17}$ and we use a fixed penalization parameter $\epsilon_n = 10^{-4}$, for all $n \in \mathbb{N}$. After 11 iterations the relative error no longer change.

The coincidence set (blue) at the left and the computed solution with the obstacle



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Torsion of an elastic-plastic prism

 $D = [0, 1] \times [0, 1]$, f(x) = -8 and $\psi(x) = -dist(x, \partial D)$. Mesh of 39216 triangles, 19865 vertices and size $h = \frac{1}{128}$. The tolerance for the stopping test is $tol = 10^{-18}$ and the penalization parameter is $\epsilon_n = 0.003$.



The computed coincidence set (blue).

The Algorithm 1 stops after 6 iterations. The algorithm presented in Xue 2004 stops after 7 iterations, while the over-relaxation algorithm with projection presented in Glowinski 1981 p. 133 stops after 93 iterations.

Bilateral elastic-plastic torsion problem

 $D = [0,1] \times [0,1], \ \psi_1(x,y) = -dist((x,y), \partial D), \ \psi_2(x,y) = 0.2$

$$g(x) = \begin{cases} 6x, & 0 < x \le 1/6, \\ 2(1-3x), & 1/6 < x \le 1/3, \\ 6(x-1/3), & 1/3 < x \le 1/2, \\ 2(1-3(x-1/3)), & 1/2 < x \le 2/3, \\ 6(x-2/3), & 2/3 < x \le 5/6, \\ 2(1-3(x-2/3)), & 5/6 < x \le 1 \end{cases}$$

and f(x, y) =

$$\begin{cases} 300, & (x,y) \in S = \{(x,y) \in D; |x-y| \le 0.1 \text{ and } x \le 0.3 \\ -70 \exp(y)g(x), & x \le 1-y \text{ and } (x,y) \notin S, \\ 15 \exp(y)g(x), & x > 1-y \text{ and } (x,y) \notin S. \end{cases}$$

We use a mesh of 9662 triangles, 4960 vertices, the size $h = \frac{1}{64}$, the tolerance for the stopping test $tol = 10^{-1}$ and the penalization parameter is $\epsilon_n = 0.003$.

Computed solution after 15 iterations



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Coincidence sets (blue) for the bottom obstacle (left) and for the top obstacle (right)





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In Karkkainen 2003, an augmented lagrangian active set strategy is employed. At each iteration, a reduced linear system associated with the inactive set is solved. Without multigrid nested iterations, the algorithm stops after 30 iterations.

In Wang 2008, at each iteration, linear systems associated to the complementary of the coincidence sets are solved. There are no information about the number of iterations, only on the CPU time; for example the total CPU time for the "up down" algorithm is 916 s and the compute time of a linear system is 46 s.

For instance, the CPU time is 15.12 s on a PC with Intel i5 2.53 GHz and 4 Go RAM in our algorithm.

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Parabolic obstacle problem

$$\begin{split} \psi \in L^2\left(0, T; H^2(D)\right) \cap H^1\left(0, T; L^2(D)\right), \\ \psi(t, x) &\leq 0 \text{ a.e. on } [0, T] \times \partial D, \ f \in L^2\left([0, T] \times D\right) \\ & \frac{\partial y}{\partial t} - \Delta y + \beta(y - \psi) \ni f \text{ a.e. in } [0, T] \times D \end{split} \tag{16}$$

together with (19).

$$\mathcal{K}(t) = \left\{ v \in H_0^1(D); \ v(x) \ge \psi(t, x) \text{ a.e. } D \right\},$$
(17)
$$\int_D \frac{\partial y}{\partial t} (y - v) dx + \int_D \nabla y \cdot \nabla (y - v) dx \le \int_D f(y - v) dx,$$

$$\forall v \in L^2 \left(0, T; H_0^1(D) \right), \ v(t) \in \mathcal{K}(t) \text{ a.e. } [0, T].$$
(18)
$$y(0, x) = y_0(x) \in \mathcal{K}(0) \subset H_0^1(D) \text{ a.e. in } D,$$
(19)

Algorithm 2.1

1) Choose n = 0, $\epsilon_0 > 0$, $y_{-1}(t, x) = y_0(x)$, $\tilde{y}_{-1}(t, x) = y_0(x)$. 2) Compute $y_n \in L^2(0, T; H^2(D)) \cap H^1(0, T; L^2(D))$ as solution of the linear parabolic equation

$$\frac{\partial y_n}{\partial t} - \Delta y_n + \left[\beta'_{\epsilon_n}(y_{n-1} - \psi)\right](y_n - \psi) = f, \text{ a.e. in } [0, T] \times D,$$
$$y_n(0, x) = y_0(x) \text{ in } D.$$

3) $\tilde{y}_n = \max(y_n, \psi)$, $\epsilon_{n+1} = \frac{1}{2}\epsilon_n$ 4) If $\|\tilde{y}_n - \tilde{y}_{n-1}\|_{L^2([0,T] \times D)} < tol$ then STOP; else n=n+1 GO TO step 2.

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Algorithm 2.2

0) Put
$$y^0(x) = y_0(x)$$
.
For $k = 0, 1, ..., m-1$, $\Delta t = T/m$ do step 1) to step 4)
1) Choose $n = 0$, $\epsilon_0 > 0$, $\tilde{y}_{-1}^{k+1} = y^k$
2) Compute $y_n^{k+1} \in H_0^1(D)$ as solution of the linear parabolic equation

$$\frac{y_n^{k+1} - y^k}{\Delta t} - \Delta y_n^{k+1} + \left[\beta_{\epsilon_n}'(y_{n-1}^{k+1} - \psi)\right](y_n^{k+1} - \psi) = f^{k+1}$$

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3)
$$\tilde{y}_{n}^{k+1} = \max(y_{n}^{k+1}, \psi), \ \epsilon_{n+1} = \frac{1}{2}\epsilon_{n}$$

4) If $\left\| \tilde{y}_{n}^{k+1} - \tilde{y}_{n-1}^{k+1} \right\|_{L^{2}(D)} < tol$ then $y^{k+1} = \tilde{y}_{n}^{k+1}$, STOP;
else n=n+1 GO TO step 2.

Two-phase Stefan problem

$$\frac{\gamma(y^{k+1}) - \gamma(y^k)}{\Delta t} - \Delta y^{k+1} = f^{k+1} \text{ in } D, \qquad (20)$$

The maximal monotone enthalpy graph $\gamma \subset \mathbb{R} \times \mathbb{R}$

$$\gamma(r) = \begin{cases} ar, & r < 0, \\ [0, L], & r = 0, \\ br + L, & r > 0 \end{cases}$$
(21)

with a, b, L positive constants related to the thermal conductivity in the liquid/solid phases and to the latent heat. The subsequent regularization γ_{ϵ} is similar to the Yosida regularization and Lipschitzian of rang $\frac{1}{\epsilon}$:

$$\gamma_{\epsilon}(r) = \begin{cases} ar, & r \leq 0, \\ \frac{1}{\epsilon}r, & 0 < r < \frac{L\epsilon}{1-\epsilon b}, \\ br + L, & r \geq \frac{L\epsilon}{1-\epsilon b}. \end{cases}$$
(22)

Algorithm 2.3

0) Choose $y^0(x) = y_0(x)$, $\epsilon > 0$. For $k = 0, 1, \dots, m-1$ do step 1) to step 3) 1) Choose n = 0, $y_{-1}^{k+1} = y^k$. 2) Compute $y_n^{k+1} \in H_0^1(D)$ as solution of the linear parabolic equation

$$\frac{\widetilde{\gamma}_{\epsilon}(y_n^{k+1}) - \gamma_{\epsilon}(y^k)}{\Delta t} - \Delta y_n^{k+1} = f^{k+1}, \text{ in } D$$

where

$$\widetilde{\gamma}_{\epsilon}(y_{n}^{k+1}(x)) = \begin{cases} a y_{n}^{k+1}(x), & y_{n-1}^{k+1}(x) \leq 0, \\ \frac{1}{\epsilon} y_{n}^{k+1}(x), & 0 < y_{n-1}^{k+1}(x) < \frac{L\epsilon}{1-\epsilon b}, \\ b y_{n}^{k+1}(x) + L, & y_{n-1}^{k+1}(x) \geq \frac{L\epsilon}{1-\epsilon b}. \end{cases}$$

3) If $\|y_n^{k+1} - y_{n-1}^{k+1}\|_{L^2(D)} < tol$ then $y^{k+1} = y_n^{k+1}$, STOP; else n=n+1 GO TO step 2.

Convergence for Algorithms 2.1 and 2.3

Proposition We have $\{y_n\}$ bounded in $L^{\infty}(0, T; L^2(D)) \cap L^2(0, T; H_0^1(D))$ and $y_n \to \widetilde{y}$ on a subsequence weakly in $L^{\infty}(0, T; L^2(D)) \cap L^2(0, T; H_0^1(D))$. Moreover, $\widetilde{y} \leq y$ a.e. in $[0, T] \times D$ and

$$\int_{Q_n} (y_n - \psi)^2 d\mathsf{x} dt \le C\epsilon_n$$

with *C* independent of *n* and $Q_n = \{(t, x) \in [0, T] \times D; y_{n-1}(t, x) < \psi(t, x)\}.$ **Proposition** i) The sequence $\{y_n^{k+1}\}$ is bounded in $H_0^1(D)$ with respect to *n*. If $y_0 \in H_0^1(D) \cap L^p(D)$ and *f* is in $\mathcal{C}(0, T; L^p(D))$, $p \ge 2$, then $\{y_n^{k+1}\}$ is bounded in $W^{2,p}(D) \cap H_0^1(D)$ with respect to *n*.

ii) If $y_n^{k+1} o \widetilde{y}^{k+1}$ weakly in $W^{2,p}(D)$, p > d, then \widetilde{y}^{k+1} satisfies

$$\frac{\gamma_{\epsilon}(\widetilde{y}^{k+1}) - \gamma_{\epsilon}(y^{k})}{\Delta t} - \Delta \, \widetilde{y}^{k+1} = f^{k+1}, \text{ in } D.$$
(23)

One phase Stefan problem

 $D = (-1, 1) \times (-1, 1)$, T = 0.5, $\psi = 0$ and f = -2 on $[0, T] \times D$ penalization parameter $\epsilon_n = 10^{-3}$, time step $\Delta t = 0.05$, mesh size h = 1/160

The Algorithm 2.1 ends after n = 10 iterations, and the Algorithm 2.2 performs n = 6 or n = 5 iterations by time step.



The free boundary position at t = 0.05, t = 0.15, t = 0.30, t = 0.5 (from the exterior to the center) at the left and computed solution at t = 0.5 at the right

Two phase Stefan problem $D = \{(x_1, x_2); x_1^2 + x_2^2 \le 1\}, T = 0.5,$ $f(t, x_1, x_2) = \begin{cases} 8(2e^{-2t} - 1), & \sqrt{x_1^2 + x_2^2} > e^{-t}, \\ 2(2e^{-2t} - 2), & \sqrt{x_1^2 + x_2^2} \le e^{-t}, \end{cases}$ $\gamma(r) = \begin{cases} r, & r < 0, \\ [0, 2], & r = 0, \\ 4r + 2, & r > 0 \end{cases}$

The exact solution is

$$y(t, x_1, x_2) = \left\{ egin{array}{ll} 2(x_1^2 + x_2^2 - e^{-2t}), & \sqrt{x_1^2 + x_2^2} > e^{-t}, \ x_1^2 + x_2^2 - e^{-2t}, & \sqrt{x_1^2 + x_2^2} \le e^{-t}. \end{array}
ight.$$

mesh of 85030 triangles and 42866 vertices The error between the exact and the calculated solution $\|y_{calc} - y_{ext}\|_{L^2(0,T;L^2(D))}$ is 0.193753 for the time step $\Delta t = 0.01$ and the number of time steps N = 50.

Two phase Stefan problem



Free boundary position at t = 0.1, t = 0.2, t = 0.3 (from the exterior to the center) at the left and computed solution at t = 0.1 at the right

Thank you!

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